A NEGATIVE SOLUTION TO THE COMPLEMENTED SUBSPACE PROBLEM FOR BANACH SPACES WITH UNCONDITIONAL BASES

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Abstract. We give a negative solution to the complemented subspace problem for Banach spaces with unconditional bases over both the real and complex fields. For every $\rho > 0$, we construct a projection P_ρ of norm less than $1 + \rho$ on a separable superreflexive space

$$X_\rho = \left(\bigoplus_{j=1}^{\infty} \ell_{p_j}^{N_j} \right)_2, \qquad p_j \downarrow 2,$$

such that $Z_\rho = P_\rho(X_\rho)$ and its dual Z_ρ^* have Schauder bases but admit no unconditional bases. Over the real field, both spaces have Gordon–Lewis local unconditional structure (GL-lust) but fail Dubinsky–Pełczyński–Rosenthal local unconditional structure (DPR-lust), disproving a conjecture of Figiel, Johnson and Tzafriri. In particular, neither is isomorphic to a Banach lattice, giving a negative solution to the separable Banach-lattice complemented subspace problem. A modification of the construction also shows that the class of separable real Banach lattices is not primary. A Lean 4 formalisation of the main results accompanies the paper.

Companion Lean 4 formalisation: `https://banach-complemented-subspaces.github.io/`

Contents

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1. Introduction

The *complemented subspace problem* (CSP) asks whether every complemented subspace of a space in a given class is isomorphic to a space in the same class. Classical instances include Pełczyński's questions for L_1-spaces and $C(K)$-spaces, posed in 1960 [27, p. 210].

The question for spaces with an unconditional basis, traditionally attributed to Banach [8], is also recorded as Problem 1.d.4 in Lindenstrauss and Tzafriri's monograph [23, p. 27]. One approach is to study whether an unconditional basis of a direct sum can be partitioned into subsequences spanning copies of its summands. Such splitting results were obtained, under additional hypotheses, by Edelstein and Wojtaszczyk [11] and later by Wojtaszczyk [32].

The corresponding problem for Banach lattices is closely related: both L_1-spaces and $C(K)$-spaces are Banach lattices, and a real space with an unconditional basis can be equivalently renormed as a Banach lattice under the coordinate order.

Plebanek and Salguero-Alarcón [29] gave a negative answer to Pełczyński's $C(K)$-space question. They constructed a 1-complemented subspace PS_2 of a $C(K)$-space which is not isomorphic to any $C(K)$-space. De Hevia, Martínez-Cervantes, Salguero-Alarcón and Tradacete [8] subsequently proved that PS_2 is not isomorphic to any Banach lattice, giving a negative solution to the Banach-lattice problem as well.

The space PS_2 is nonseparable, so these results do not settle the problem for separable Banach lattices. For a fuller account of the CSP and its history, we refer to the survey of de Hevia and Tradacete [9].

We give a negative solution to the CSP for unconditional bases over both the real and complex fields, and for separable real Banach lattices. The examples are complemented subspaces of superreflexive spaces with a 1-unconditional basis, and the projections have norms arbitrarily close to one. Closely related mixed sums of finite-dimensional ℓ_p-spaces were used by Figiel in his classical construction of a reflexive Banach space not isomorphic to its Cartesian square [12]. Our conclusions follow from separating two notions of local unconditional structure.

Roughly, the Dubinsky–Pełczyński–Rosenthal version (DPR-lust) [10] asks that each finite-dimensional subspace lie in a larger finite-dimensional subspace with a uniformly controlled unconditional basis. The Gordon–Lewis version (GL-lust) [14] instead uses uniformly bounded factorisations through spaces with unconditional bases, which need not be subspaces of the given space. Thus DPR-lust implies GL-lust, but whether the converse holds was unknown [9]; Figiel, Johnson and Tzafriri already emphasised the distinction between the two notions [13, Remark 2.3].

Every Banach lattice has DPR-lust, while every complemented subspace of a Banach lattice has GL-lust. The main conjecture of Figiel, Johnson and Tzafriri [13, p. 396] states that such complemented subspaces also have DPR-lust. The following theorem disproves this conjecture and, in particular, separates GL-lust from DPR-lust in separable superreflexive spaces.

Theorem A. *For every $\rho > 0$, there exist sequences $(N_j)_{j=1}^\infty$ in $\mathbb{N}$ and $(p_j)_{j=1}^\infty$ in $(2,3]$, with $p_j \downarrow 2$, and a projection P_ρ on the separable superreflexive real Banach lattice*

$$X_\rho = \left(\bigoplus_{j=1}^\infty \ell_{p_j}^{N_j}(\mathbb{R}) \right)_2$$

such that, with $Z_\rho = P_\rho(X_\rho)$, we have:

(i) *For $V \in \{Z_\rho, Z_\rho^*\}$,*

$$\chi_{\mathrm{GL}}(V) \le \|P_\rho\| < 1 + \rho, \qquad \chi_{\mathrm{DPR}}(V) = \infty.$$

(ii) *The projection P_ρ can be chosen so that $\|I_{X_\rho} - P_\rho\| < 1 + \rho$ and (i) also holds for $(I_{X_\rho} - P_\rho)(X_\rho)$ and its dual, with P_ρ replaced by $I_{X_\rho} - P_\rho$.*

It follows that DPR-lust is not inherited by complemented subspaces. Since every Banach lattice has DPR-lust and this property is invariant under isomorphism, neither Z_ρ nor Z_ρ^* is isomorphic to a Banach lattice. In particular, neither space admits an unconditional basis. Together with the complex construction, this gives the following negative solutions to the CSP for unconditional bases and for separable real Banach lattices.

Theorem B. *For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and every $\rho > 0$, there are a Banach space X_ρ over $\mathbb{K}$ with a 1-unconditional basis and a projection P_ρ on X_ρ with $\|P_\rho\| < 1 + \rho$ such that, writing $Z_\rho = P_\rho(X_\rho)$, neither Z_ρ nor Z_ρ^* admits an unconditional basis. If $\mathbb{K} = \mathbb{R}$, neither space is isomorphic to a Banach lattice.*

The spaces Z_ρ and Z_ρ^* are superreflexive and admit shrinking, boundedly complete bases, by Corollary 5.2(i). By Corollary 5.2(ii), these bases can be chosen to be asymptotically monotone.

The projection bound is sharp over $\mathbb{C}$. Kalton and Wood [18] showed that every contractively complemented subspace of a complex Banach space with a 1-unconditional basis again admits a 1-unconditional basis; see also [30, Theorem 7.18]. Thus the complex example satisfies $1 < \|P_\rho\| < 1 + \rho$, with ρ arbitrarily small. The corresponding 1-unconditional conclusion fails over $\mathbb{R}$: counterexamples were obtained by Lewis [22] and by Benyamini, Flinn and Lewis [3]; see also [30, Proposition 7.25].

De Hevia and Tradacete also asked whether, in a decomposition of a Banach lattice into two closed subspaces, at least one must be isomorphic to a Banach lattice [9, Question 4]. This is the *primariness* question for the class; see Section 4.3.

In earlier work [1], we gave a negative answer by decomposing a nonseparable $C(K)$-space into two subspaces neither isomorphic to a Banach lattice. Theorem A(ii) gives the following separable counterpart.

Corollary C. *The class of separable real Banach lattices is not primary.*

The earlier decomposition also shows that the class of $C(K)$-spaces is not primary [1]. In the present setting, Theorem A(ii) gives the same conclusion for separable real spaces with DPR-lust and for those with an unconditional basis.

All four spaces in Theorem A(ii) also have a contractive unconditional finite-dimensional decomposition and the metric approximation property, as shown in Corollary 4.5(ii). The Kalton–Peck space has an unconditional decomposition into two-dimensional subspaces [17], while Johnson, Lindenstrauss and Schechtman [15] showed that it fails GL-lust. Casazza and Kalton [5, Theorem 3.8] proved that GL-lust together with an unconditional finite-dimensional decomposition of uniformly bounded block dimensions forces an unconditional basis. Thus the unconditional finite-dimensional decompositions in the present examples necessarily have unbounded block dimensions.

Over $\mathbb{R}$, Lacey's stabilisation theorem [21, p. 49, second Corollary] states that a Banach space V has GL-lust if and only if $V \oplus c_0$ has DPR-lust. Consequently, $Z_\rho \oplus_\infty c_0$ has DPR-lust, although its contractively complemented subspace Z_ρ does not. Thus DPR-lust fails to pass even to contractively complemented subspaces.

Idea of the proof and organisation. Section 2 introduces the notation and recalls the two notions of local unconditional structure, together with the Hilbert-space and probability conventions used in the proofs.

The construction in Section 3 forms the technical core of the paper. It has two main steps: constructing an ℓ_2-sum $Y = (\bigoplus_{j=1}^\infty E_j)_2$ without DPR-lust, and realising an isomorphic copy Z_ρ as a complemented subspace of X_ρ with projection norm less than $1 + \rho$. Both use the finite-dimensional tensor spaces $E_j \subseteq L_{p_j}$ introduced in Section 3.1, with suitably chosen exponents $p_j \downarrow 2$.

To prove failure of DPR-lust, we must exclude uniformly controlled unconditional bases in every finite-dimensional superspace of E_j inside Y. The tensor and fourth-moment estimates in Lemmas 3.1 and 3.4 first give a more restricted obstruction: a subspace of dimension at most $3 \dim E_j$ with a controlled unconditional basis has small Hilbert overlap with E_j, even after a Hilbert space is adjoined, as shown in Theorem 3.5 and proposition 3.6.

The crucial passage to arbitrary superspaces is the finite selection principle of Lemma 3.8. Using the trace estimate of Lemma 3.2, it selects at most $3 \dim E_j$ coordinates from a proposed basis with a controlled unconditional constant, so that the projection onto those coordinates retains a substantial trace after compression to E_j. The number of selected coordinates is independent of the dimension of the superspace.

In Section 3.5.3, we recursively choose the subsequence $(E_j)_{j=1}^{\infty}$ of tensor spaces used to form Y. Together with Lemma 3.7, this choice allows the selected complementary components to be replaced by vectors in a Hilbert space with controlled distortion. The Gaussian estimate then turns the retained trace into a lower bound for overlap. Comparing the two overlap bounds forces the unconditional constant to grow with j, uniformly over all such superspaces. This gives Theorem 3.9 and proves that Y fails DPR-lust.

Complementation comes from tensorising the projections onto the two-dimensional function spaces defining E_j, as explained in Section 3.1. Near $p = 2$, the first-order terms in the logarithm of the one-factor projection bound cancel. The choice of parameters in Section 3.4 therefore makes the tensor projection norms tend to one while the obstruction to DPR-lust grows. The complemented embedding is then constructed in Section 3.6.3.

The embeddings in Section 3.6 pass to finite-dimensional ℓ_p spaces while retaining the required projection bound. Over $\mathbb{R}$, Lemma 3.10 uses exact covariance on a circle grid; over $\mathbb{C}$, Lemma 3.11 uses finite conditional expectations and a correction of an approximate inverse. The embeddings and their left inverses are uniformly bounded. The corresponding diagonal operators $A\colon Y \to X_\rho$ and $B\colon X_\rho \to Y$ satisfy $BA = I_Y$, so $Z_\rho = A(Y)$ is isomorphic to Y and $P_\rho = AB$ is a projection of norm less than $1+\rho$. The ambient sum X_ρ has its canonical 1-unconditional basis.

In Section 4, we establish the GL/DPR separation and its dual consequences. The GL-lust bound follows from Lemma 4.1, while the quotient version of the local Hilbert estimate yields the dual obstruction in Proposition 4.3. Over $\mathbb{R}$, a stronger recursion controlling the earlier ambient spaces gives two complementary summands, neither isomorphic to a Banach lattice, as established in Corollary 4.5.

Finally, Section 5 constructs conditional asymptotically monotone bases for the examples and their duals, as stated in Corollary 5.2. The section concludes with brief geometric consequences of the ℓ_2-sum structure.

2. Preliminaries and notation

We follow the notation and terminology in [23], unless otherwise specified. We write $\mathbb{K}$ for the scalar field, which may be either $\mathbb{R}$ or $\mathbb{C}$. Unless stated otherwise, the definitions below apply over both fields. We treat the two scalar fields together in Section 3, indicating where their proofs differ.

2.1. **Notation and unconditional constants.** Throughout, an *operator* is a bounded linear map. For Banach spaces E and F over $\mathbb{K}$, we denote by $\mathcal{B}(E, F)$ the Banach space of operators from E to F, equipped with the operator norm. When $E = F$, we write $\mathcal{B}(E)$ in place of $\mathcal{B}(E, E)$. The dual of E is denoted by E^*, and I_E denotes the identity operator on E. Given $x \in F$ and $f \in E^*$, we write $x \otimes f$ for the rank-one operator

$$(x \otimes f)(y) = f(y)x \qquad (y \in E).$$

We write $\mathbb{N} = \{1, 2, \dots\}$. For a family $(E_j)_{j\in J}$ of Banach spaces, the notation $(\bigoplus_{j\in J} E_j)_2$ denotes their *ℓ_2-sum*: its elements are the families $(x_j)_{j\in J}$, with $x_j \in E_j$ for $j \in J$ and

$$\|(x_j)_{j\in J}\| = \left(\sum_{j\in J} \|x_j\|_{E_j}^2\right)^{1/2} < \infty.$$

Let $(b_i)_{i\in I}$ be a Schauder basis of a Banach space E. We use $I = \{1, \dots, d\}$ when E is finite-dimensional and $I = \mathbb{N}$ otherwise. We denote its coordinate functionals by $(b_i^*)_{i\in I} \subseteq E^*$. Its *basis constant* is

$$b\big((b_i)_{i\in I}\big) = \sup_{m\in I} \left\| \sum_{i=1}^m b_i \otimes b_i^* \right\|,$$

the supremum of the norms of its initial coordinate projections. When $I = \mathbb{N}$, the basis is *asymptotically monotone* if

$$\lim_{m\to\infty} \left\| \sum_{i=1}^m b_i \otimes b_i^* \right\| = 1;$$

see [24, Section 1].

For $C \geq 1$, we say that the basis is *C-unconditional* if

$$\left\| \sum_{i\in I} \theta_i b_i \otimes b_i^* \right\| \leq C$$

for every finitely supported family $(\theta_i)_{i\in I}$ in $\mathbb{K}$ satisfying $|\theta_i| \leq 1$ for every $i \in I$. In particular, the coordinate projection $\sum_{i\in M} b_i \otimes b_i^*$ has norm at most C for every $M \subseteq I$.

For a finite-dimensional Banach space E, we denote by $u(E)$ the infimum of the unconditional basis constants over all bases of E. We shall use without further mention that rescaling the basis vectors by nonzero scalars leaves the unconditional basis constant unchanged.

2.2. **Hilbert norms and factorisation.** In addition to their L_p norms, the tensor spaces of Section 3.1 carry the Euclidean norm of their coefficient vectors; we call this the *coefficient Hilbert norm* and denote it by $|\cdot|_2$. Hilbert tensor products are denoted by $\otimes_2$. Whenever we discuss adjoints, orthogonality or Hilbert–Schmidt norms on these spaces, we refer to their coefficient Hilbert structures. In particular, orthogonal projections need not be contractive for the Banach norm.

Our inner products are linear in the first variable. We write A^T for the transpose of a real matrix and A^* for a Hilbert adjoint. For Hilbert spaces H and K with $\dim H = d < \infty$, we set

$$\langle A, B\rangle_{\mathrm{HS}} = \operatorname{tr}(B^*A), \qquad \|T\|_{\mathrm{HS}}^2 = \sum_{i=1}^d |Te_i|_2^2,$$

for $A, B, T \in \mathcal{B}(H, K)$, where $(e_i)_{i=1}^d$ is any orthonormal basis of H. The notation $\|A\|_{2\to 2}$ refers to the operator norm between the specified Hilbert spaces.

For isomorphic Banach spaces E and F, we denote their *Banach–Mazur distance* by

$$d_{\mathrm{BM}}(E, F) = \inf\{\|J\|\|J^{-1}\| : J\colon E \to F \text{ is an isomorphism}\}.$$

Given $D \geq 1$, we say that E is *D-Hilbertian* if there is a Hilbert norm $|\cdot|_H$ on E such that $|x|_H \leq \|x\|_E \leq D|x|_H$ for every $x \in E$. Equivalently, E admits an isomorphism J onto a Hilbert space with $\|J\| \leq 1$ and $\|J^{-1}\| \leq D$.

For $T \in \mathcal{B}(E, F)$, we define its *Hilbert-factorisation norm* by

$$\gamma_2(T) = \inf\{\|A\|\|B\| : T = BA,\ A \in \mathcal{B}(E, H),\ B \in \mathcal{B}(H, F),\ H \text{ a Hilbert space}\},$$

with the convention that the infimum of the empty set is $+\infty$. The operators for which γ_2 is finite form a linear space on which γ_2 is a norm. We shall also use that $\gamma_2(STR) \leq \|S\|\,\gamma_2(T)\|R\|$.

2.3. **Local unconditional structure.** We shall use two standard notions of local unconditional structure: DPR-lust and GL-lust. They differ in whether the finite-dimensional spaces with unconditional bases must lie inside the given space or may instead be auxiliary spaces in a factorisation. We now recall the definitions for a Banach space Z.

We say that Z has *local unconditional structure in the sense of Dubinsky, Pełczyński and Rosenthal*, or *DPR-lust* [10], if there is a constant $C \geq 1$ such that every nonzero finite-dimensional subspace $V \subseteq Z$ is contained in a finite-dimensional subspace $F \subseteq Z$ with a C-unconditional basis. To keep track of the constants, we put

$$\lambda_{\mathrm{DPR}}(Z;V) = \inf\{u(F) : V \subseteq F \subseteq Z,\ \dim F < \infty\},$$
$$\chi_{\mathrm{DPR}}(Z) = \sup_{\substack{V \subseteq Z \\ 0<\dim V<\infty}} \lambda_{\mathrm{DPR}}(Z;V).$$

We call $\chi_{\mathrm{DPR}}(Z)$ the *DPR constant* of Z. Thus Z has DPR-lust precisely when this constant is finite.

For the Gordon–Lewis notion [14], the finite-dimensional space with an unconditional basis may instead be an auxiliary space. Given a nonzero finite-dimensional $V \subseteq Z$, let $\iota_V : V \to Z$ denote the inclusion. We say that Z has *local unconditional structure in the sense of Gordon and Lewis*, or *GL-lust*, if each ι_V factors through a finite-dimensional space U with a 1-unconditional basis, with the products of the norms of the two operators bounded independently of V. The *GL constant* is

$$\chi_{\mathrm{GL}}(Z) = \sup_{\substack{V \subseteq Z \\ 0<\dim V<\infty}} \inf\{\|a\|\|b\| : V \xrightarrow{a} U \xrightarrow{b} Z,\ ba = \iota_V\}.$$

Here the infimum is taken over all finite-dimensional Banach spaces U with a 1-unconditional basis and all bounded linear maps $a\colon V \to U$ and $b\colon U \to Z$ satisfying the indicated identity. Thus Z has GL-lust precisely when $\chi_{\mathrm{GL}}(Z) < \infty$. The map b need not be injective; unlike the superspace F in the DPR definition, U need not be realised inside Z.

Every finite-dimensional space with a C-unconditional basis admits an equivalent norm, at distortion at most C, for which that basis is 1-unconditional. Applying this observation to the superspaces in the DPR definition gives $\chi_{\mathrm{GL}}(Z) \leq \chi_{\mathrm{DPR}}(Z)$. Both properties are invariant under isomorphism, by applying an isomorphism to the superspaces or the factorisations, respectively. The infima need not be attained, so we allow an arbitrarily small increase in a constant when choosing a superspace or a factorisation.

For every nonzero real Banach lattice L, one has $\chi_{\mathrm{DPR}}(L) = 1$ and hence $\chi_{\mathrm{GL}}(L) = 1$ [13, p. 397].

2.4. **Decompositions and approximation.** A sequence $(Z_j)_{j=1}^{\infty}$ of finite-dimensional subspaces of a Banach space Z is a *finite-dimensional decomposition* if every $z \in Z$ has a unique norm-convergent representation $z = \sum_{j=1}^{\infty} z_j$, with $z_j \in Z_j$ for every $j \geq 1$. We call this decomposition *contractive and unconditional* if, for every $m \geq 1$ and every choice of vectors $(z_j)_{j=1}^{m}$ and scalars $(\theta_j)_{j=1}^{m}$ satisfying $z_j \in Z_j$ and $|\theta_j| \leq 1$ for $1 \leq j \leq m$, one has

$$\left\| \sum_{j=1}^{m} \theta_j z_j \right\| \leq \left\| \sum_{j=1}^{m} z_j \right\|.$$

Here the multipliers act on the summands of the decomposition. The condition places no uniform bound on the unconditional basis constants of the spaces Z_j.

The space Z has the *metric approximation property* if its identity operator can be approximated uniformly on compact subsets by finite-rank contractions. Clearly, a space with a contractive unconditional finite-dimensional decomposition has the metric approximation property, since its initial block projections are finite-rank contractions converging strongly to the identity.

2.5. **Probability conventions.** The tensor spaces of Section 3.1 are defined on probability spaces. For $1 < p < \infty$, we write $p' = p/(p-1)$ for the conjugate exponent. Finite-dimensional ℓ_p spaces carry the usual counting norm unless a normalised probability norm is explicitly specified.

For an integrable random variable ξ, its expectation is denoted by $\mathbb{E}[\xi]$. A subscript indicates the variables over which the expectation is taken: for example, $\mathbb{E}_t[\Phi(t,s)]$ means that t is averaged and s is held fixed. In the absence of a subscript, we average over all random variables occurring in the expression. By a sequence of *Rademacher variables* $(\varepsilon_i)_{i=1}^{\infty}$ we mean independent random variables taking the values -1 and 1, each with probability 1/2.

A *standard scalar Gaussian* is a standard real Gaussian when $\mathbb{K} = \mathbb{R}$, and a variable $(g+ih)/\sqrt{2}$, with g, h independent standard real Gaussians, when $\mathbb{K} = \mathbb{C}$. In both cases its second absolute moment is one.

Let H be a Hilbert space over $\mathbb{K}$ of dimension $q \geq 1$, and let $(e_i)_{i=1}^{q}$ be an orthonormal basis of H. A *normalised Gaussian vector* in H is a random vector of the form $\zeta = q^{-1/2}\sum_{i=1}^{q} g_i e_i$, where $(g_i)_{i=1}^{q}$ are independent standard scalar Gaussian variables. Its covariance operator is I_H/q. Consequently, for every linear operator T from H into a Hilbert space,

$$\mathbb{E}[|\zeta|_2^2] = 1, \qquad \|T\|_{\mathrm{HS}}^2 = q\,\mathbb{E}[|T\zeta|_2^2]. \tag{2.1}$$

3. Construction of the examples

We work over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ unless stated otherwise, indicating the differences between the two cases where necessary. Fix $\rho > 0$, and write X, P, Z for X_ρ, P_ρ, Z_ρ.

We construct an ℓ_2-sum of finite-dimensional spaces without DPR-lust and realise it as a complemented subspace of a space with a 1-unconditional basis, with projection norm less than $1+\rho$.

3.1. **Circle and sphere tensors.** For $\mathbb{K} = \mathbb{R}$, let $\Omega = \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ carry normalised Haar measure and put $w(t) = \sqrt{2}(\cos t, \sin t)$. For $\mathbb{K} = \mathbb{C}$, let Ω be the unit sphere of $\mathbb{C}^2$ with its invariant probability measure and put $w(t) = \sqrt{2}t$. In either case, w is uniformly distributed on the sphere of radius $\sqrt{2}$ in $\mathbb{K}^2$. For each $n \geq 1$, set

$$\mathcal{H}_n = (\mathbb{K}^2)^{\otimes_2 n}, \qquad q_n = \dim \mathcal{H}_n = 2^n.$$

For $t = (t_s)_{s=1}^n \in \Omega^n$, the tensor $W(t) = \bigotimes_{s=1}^n w(t_s)$ satisfies

$$|W(t)|_2^2 = q_n, \qquad \mathbb{E}_t[W(t)W(t)^*] = I_{\mathcal{H}_n}. \tag{3.1}$$

For $2 < p \leq 3$, we denote by $E(n,p)$ the space $\mathcal{H}_n$ with norm

$$\|x\|_{E(n,p)} = \|f_x\|_{L_p(\Omega^n)}, \qquad f_x(t) = \langle x, W(t)\rangle.$$

We identify $E(n,p)$ with its image under $x \mapsto f_x$ in $L_p(\Omega^n)$. For $r > 0$, put

$$\alpha_r = (\mathbb{E}[|w_1|^r])^{1/r}, \qquad c_p = \alpha_p\alpha_{p'}, \qquad \gamma_p = (\mathbb{E}[|g|^p])^{1/p}, \tag{3.2}$$

where g is a standard scalar Gaussian as in Section 2.5. These constants depend on the fixed scalar field. Explicitly,

$$\alpha_r = \begin{cases} \|\sqrt{2}\cos t\|_{L_r(\mathbb{T})}, & \mathbb{K} = \mathbb{R}, \\ \sqrt{2}(1+r/2)^{-1/r}, & \mathbb{K} = \mathbb{C}, \end{cases} \tag{3.3}$$

since the squared modulus of the first coordinate of a uniform unit vector in $\mathbb{C}^2$ is uniform on $[0,1]$. Throughout this subsection, ζ denotes a normalised Gaussian vector in $\mathcal{H}_n$.

Lemma 3.1. *Let $E = E(n,p)$. Then the following hold.*

(i) *For every $x \in \mathcal{H}_n$,*

$$|x|_2 \leq \|x\|_E \leq \alpha_p^n |x|_2.$$

(ii) *If x is a tensor product of n unit vectors, then*

$$\|x\|_E = \alpha_p^n, \qquad \|\langle\,\cdot\,, x\rangle\|_{E^*} \le \alpha_{p'}^n.$$

(iii) *For the normalised Gaussian vector ζ and every contraction $A\colon \mathcal{H}_n \to \mathcal{H}_n$,*

$$\mathbb{E}_\zeta[\|\zeta\|_E^p] = \gamma_p^p, \qquad \mathbb{E}_\zeta[\|A\zeta\|_E^2] \le \gamma_p^2.$$

Proof. For (i), isotropy gives $\|f_x\|_2 = |x|_2$, which proves the lower bound. To prove the upper bound, let $V\colon \mathbb{K}^2 \to K$ be a linear map into a Hilbert space. Choose an orthonormal eigenbasis of V^*V, with corresponding eigenvalues λ_1, λ_2. By orthogonal or unitary invariance, the coordinates of w in this basis have the same joint distribution as (w_1, w_2). Hence

$$\mathbb{E}[|Vw|_2^p] = \mathbb{E}\left[(\lambda_1|w_1|^2 + \lambda_2|w_2|^2)^{p/2}\right].$$

For fixed $\lambda_1+\lambda_2$, the right-hand side is convex in the eigenvalues and therefore attains its maximum when one of them is zero. Since $\lambda_1 + \lambda_2 = \operatorname{tr}(V^*V) = \|V\|_{\mathrm{HS}}^2$, we obtain

$$\mathbb{E}[|Vw|_2^p] \le \alpha_p^p(\lambda_1+\lambda_2)^{p/2} = \alpha_p^p\|V\|_{\mathrm{HS}}^p.$$

Taking pth roots gives

$$(\mathbb{E}[|Vw(t)|_2^p])^{1/p} \le \alpha_p\|V\|_{\mathrm{HS}}.$$

For the tensor estimate, we argue by induction on n, the case $n=1$ following from the estimate above. For $n \ge 2$, write $x = e_1 \otimes x_1 + e_2 \otimes x_2$, where $(e_i)_{i=1}^2$ is the standard basis of $\mathbb{K}^2$ and $x_1, x_2 \in \mathcal{H}_{n-1}$. With $t' = (t_s)_{s=2}^n$, we have

$$f_x(t) = \overline{w_1(t_1)}f_{x_1}(t') + \overline{w_2(t_1)}f_{x_2}(t').$$

Since $\overline{w}$ has the same distribution as w, we may apply the one-variable estimate in t_1. Integrating over t' and then using Minkowski's inequality in $L_{p/2}(\Omega^{n-1})$ gives

$$\|f_x\|_{L_p(\Omega^n)} \le \alpha_p \left\| \left(\sum_{i=1}^2 |f_{x_i}|^2\right)^{1/2} \right\|_{L_p(\Omega^{n-1})} \le \alpha_p \left(\sum_{i=1}^2 \|f_{x_i}\|_{L_p(\Omega^{n-1})}^2\right)^{1/2}.$$

The induction hypothesis therefore yields

$$\|x\|_E \le \alpha_p^n \left(|x_1|_2^2 + |x_2|_2^2\right)^{1/2} = \alpha_p^n|x|_2.$$

For (ii), let x be a product unit vector. The scalar factors of f_x have the same absolute moments as the first coordinate of w, so $\|x\|_E = \alpha_p^n$. The dual estimate follows from isotropy and Hölder's inequality: $|\langle y, x\rangle| = |\mathbb{E}[f_y\overline{f_x}]| \le \|y\|_E\alpha_{p'}^n$.

To prove (iii), observe that, for each fixed t, $\langle\zeta, W(t)\rangle$ is a scalar Gaussian of variance one, so Fubini's theorem gives the identity in (iii). Replacing ζ by $A\zeta$ changes that variance to $q_n^{-1}|A^*W(t)|_2^2 \le 1$. The same moment calculation, followed by the inequality between the second and pth moments, gives the remaining estimate since $p \ge 2$. □

To complement $E(n,p)$ in $L_p(\Omega^n)$, we use the projection onto the span of $\overline{w_1}, \overline{w_2}$ in each variable. In the real case, this is the first-harmonic projection. In either field it is

$$Qf(t) = \langle \mathbb{E}_s[f(s)w(s)], w(t)\rangle.$$

The coefficient map $f \mapsto \mathbb{E}_s[f(s)w(s)]$ from $L_p(\Omega)$ to ℓ_2^2 has norm at most $\alpha_{p'}$, while the map $x \mapsto \langle x, w(\cdot)\rangle$ has norm α_p, and hence $\|Q\| \le c_p$. Fubini's theorem gives the same bound when Q acts on one variable of $L_p(\Omega^n)$. Composing these commuting projections gives a projection onto $E(n,p)$ of norm at most c_p^n. We denote this projection by P_n. In Section 3.6, we obtain a corresponding projection on a finite-dimensional ℓ_p space.

A trace estimate. The symmetries of $E(n,p)$ also allow us to control the trace of an operator in terms of its Hilbert factorisation norm.

Lemma 3.2. *Every operator T on $E(n,p)$ satisfies*

$$\frac{|\operatorname{tr} T|}{q_n} \le \frac{\gamma_p}{\alpha_p^n}\gamma_2(T). \tag{3.4}$$

Proof. Let $E = E(n,p)$, and let U be uniformly distributed over the finite group generated by sign changes and coordinate interchanges in the tensor factors. Its elements are isometries for both norms. Averaging over this group removes off-diagonal entries and equalises the diagonal entries; see [2, Section 2.5]. Thus, for any product unit vector x,

$$\mathbb{E}_U[U^{-1}TU] = \frac{\operatorname{tr} T}{q_n} I_E, \qquad \mathbb{E}_U[(Ux)(Ux)^*] = I_{\mathcal{H}_n}/q_n = \mathbb{E}[\zeta\zeta^*].$$

Let $T = BA$ be a Hilbert factorisation, with $A\colon E \to H$ and $B\colon H \to E$. Applying Lemma 3.1 and using equality of the covariances, we obtain

$$\frac{|\operatorname{tr} T|}{q_n}\alpha_p^n \overset{\text{(ii)}}{=} \left\|\mathbb{E}_U[U^{-1}BAUx]\right\|_E \le \|B\|\left(\mathbb{E}_U[|AUx|_H^2]\right)^{1/2} = \|B\|\left(\mathbb{E}_\zeta[|A\zeta|_H^2]\right)^{1/2} \overset{\text{(iii)}}{\le} \gamma_p\|A\|\|B\|.$$

Taking the infimum over Hilbert factorisations proves the estimate. □

3.2. **A fourth-moment estimate.** We shall need a fourth-moment bound for the vectors of a basis and its Hilbert biorthogonal family. We obtain it from King's multiplicativity theorem [19] and a Hilbert–Schmidt estimate.

Put $\tau = 5/4$ over $\mathbb{R}$ and $\tau = 10/9$ over $\mathbb{C}$. We define an operator on $M_2(\mathbb{K})$ and its tensor power on $M_{q_n}(\mathbb{K})$ by

$$\mathcal{D}(A) = \begin{cases} (\operatorname{tr}(A)I_{\mathbb{R}^2} + A + A^{\mathsf{T}})/4, & \mathbb{K} = \mathbb{R}, \\ (\operatorname{tr}(A)I_{\mathbb{C}^2} + A)/3, & \mathbb{K} = \mathbb{C}, \end{cases} \qquad \mathcal{D}_n = \mathcal{D}^{\otimes n}.$$

For a uniform unit vector u in $\mathbb{K}^2$, the fourth-moment identity takes the form

$$\mathbb{E}[uu^*Auu^*] = \frac{1}{2}\mathcal{D}(A). \tag{3.5}$$

Over $\mathbb{R}$, this follows from $\mathbb{E}[u_1^4] = 3/8$, $\mathbb{E}[u_1^2u_2^2] = 1/8$ and the vanishing of odd moments. Over $\mathbb{C}$, the corresponding identities are $\mathbb{E}[|u_1|^4] = 1/3$ and $\mathbb{E}[|u_1u_2|^2] = 1/6$, together with phase invariance. The map $\mathcal{D}$ is Hilbert–Schmidt self-adjoint. We next estimate the action of $\mathcal{D}_n$ on positive rank-one matrices.

Lemma 3.3. *For $b \in \mathcal{H}_n$,*

$$\|\mathcal{D}_n(bb^*)\|_{\mathrm{HS}}^2 \le (\tau/2)^n |b|_2^4.$$

Proof. For a linear map $\Phi\colon M_d(\mathbb{C}) \to M_d(\mathbb{C})$, write

$$\nu_2(\Phi) = \sup\{\|\Phi(A)\|_{\mathrm{HS}} : A \ge 0,\ \operatorname{tr} A = 1\}.$$

King's theorem [19, Theorem 1] gives $\nu_2(\Phi^{\otimes n}) = \nu_2(\Phi)^n$ for maps of the form $\Phi(A) = \sum_{\ell=1}^m \operatorname{tr}(M_\ell A)R_\ell$, where $M_\ell, R_\ell \ge 0$, $\operatorname{tr} R_\ell = 1$ for $1 \le \ell \le m$, and $\sum_{\ell=1}^m M_\ell = I_{\mathbb{C}^d}$.

In the real case, extend $\mathcal{D}$ complex-linearly to $M_2(\mathbb{C})$. Let e_1, e_2 be the standard unit vectors in $\mathbb{C}^2$, and put

$$\mathcal{U}_{\mathbb{R}} = \left\{e_1, e_2, \frac{e_1+e_2}{\sqrt{2}}, \frac{e_1-e_2}{\sqrt{2}}\right\},$$
$$\mathcal{U}_{\mathbb{C}} = \mathcal{U}_{\mathbb{R}} \cup \left\{\frac{e_1+ie_2}{\sqrt{2}}, \frac{e_1-ie_2}{\sqrt{2}}\right\}.$$

With $P_u = uu^*$, the defining formulas give

$$\mathcal{D}(A) = \frac{2}{|\mathcal{U}_{\mathbb{K}}|}\sum_{u\in\mathcal{U}_{\mathbb{K}}} \operatorname{tr}(P_uA)P_u, \qquad \frac{2}{|\mathcal{U}_{\mathbb{K}}|}\sum_{u\in\mathcal{U}_{\mathbb{K}}} P_u = I_{\mathbb{C}^2}.$$

Thus King's theorem applies. For positive A with $\operatorname{tr} A = 1$, we have $\operatorname{tr} \mathcal{D}(A) = 1$, and the eigenvalues of $\mathcal{D}(A)$ lie in $[1/4, 3/4]$ when $\mathbb{K} = \mathbb{R}$ and in $[1/3, 2/3]$ when $\mathbb{K} = \mathbb{C}$. Consequently $\nu_2(\mathcal{D})^2 = \tau/2$, with equality at $A = e_1 e_1^*$. Multiplicativity and homogeneity complete the proof. □

We now use this estimate to bound the fourth moment of a square function associated with two families of tensors.

Lemma 3.4. *Let $B, V : \ell_2^k(\mathbb{K}) \to \mathcal{H}_n$ have columns $(b_i)_{i=1}^k$ and $(g_i)_{i=1}^k$, respectively. Let x be the tensor product of n independent uniform unit vectors in $\mathbb{K}^2$, independent of a uniform $t \in \Omega^n$, and set*

$$S(x,t)^2 = \sum_{i=1}^k |\langle x, g_i\rangle|^2 |f_{b_i}(t)|^2.$$

Then

$$\mathbb{E}[S^2] = q_n^{-1} \sum_{i=1}^k |b_i|_2^2 |g_i|_2^2, \tag{3.6}$$

$$\mathbb{E}[S^4] \le \frac{k}{q_n} (\|B\|\|V\|)^4 \tau^n. \tag{3.7}$$

The norms of B and V are their Hilbert operator norms.

Proof. The covariances $I_{\mathcal{H}_n}/q_n$ and $I_{\mathcal{H}_n}$ of x and $W(t)$ give, for $i = 1, \dots, k$,

$$\mathbb{E}_x[|\langle x, g_i\rangle|^2] = q_n^{-1}|g_i|_2^2, \qquad \mathbb{E}_t[|f_{b_i}(t)|^2] = |b_i|_2^2.$$

Since x and t are independent, the expectation of each summand in S^2 is the product of these two expectations. Summing over $i = 1, \dots, k$ proves the first identity.

For the fourth-moment bound, let $R_i = b_i b_i^*$ for $i = 1, \dots, k$. For any matrix A,

$$\sum_{i=1}^k |\langle R_i, A\rangle_{\mathrm{HS}}|^2 = \sum_{i=1}^k |(B^*AB)_{ii}|^2 \le \|B\|^4 \|A\|_{\mathrm{HS}}^2.$$

Applying this inequality with $A = \mathcal{D}_n(R_j)$, summing over $j = 1, \dots, k$, and using Lemma 3.3, we obtain

$$\sum_{i,j=1}^k |\langle R_i, \mathcal{D}_n(R_j)\rangle_{\mathrm{HS}}|^2 \le \|B\|^4 (\tau/2)^n \sum_{j=1}^k |b_j|_2^4 \le k\|B\|^8 (\tau/2)^n. \tag{3.8}$$

The same estimate holds for $g_i g_i^*$ with V in place of B. To relate these estimates to S, we apply (3.5) in each tensor factor, which gives

$$\mathbb{E}_x\left[|\langle x, g_i\rangle|^2 |\langle x, g_j\rangle|^2\right] = q_n^{-1} \langle g_i g_i^*, \mathcal{D}_n(g_j g_j^*)\rangle_{\mathrm{HS}}.$$

Since $W(t)$ has the distribution of $\sqrt{q_n}$ times an independent copy of x,

$$\mathbb{E}_t\left[|f_{b_i}(t)|^2 |f_{b_j}(t)|^2\right] = q_n \langle R_i, \mathcal{D}_n(R_j)\rangle_{\mathrm{HS}}.$$

Expanding the square of the sum defining S^2 and using independence of x and t, the preceding identities give

$$\begin{aligned}
\mathbb{E}[S^4] &= \sum_{i,j=1}^k \langle R_i, \mathcal{D}_n(R_j)\rangle_{\mathrm{HS}} \langle g_i g_i^*, \mathcal{D}_n(g_j g_j^*)\rangle_{\mathrm{HS}} \\
&\le \left(\sum_{i,j=1}^k |\langle R_i, \mathcal{D}_n(R_j)\rangle_{\mathrm{HS}}|^2\right)^{1/2} \left(\sum_{i,j=1}^k |\langle g_i g_i^*, \mathcal{D}_n(g_j g_j^*)\rangle_{\mathrm{HS}}|^2\right)^{1/2} \\
&\le k(\|B\|\|V\|)^4 (\tau/2)^n.
\end{aligned}$$

Here the factors q_n^{-1} and q_n cancel, and the inequalities follow successively from Cauchy–Schwarz and the bounds in (3.8) for the columns of B and V. Since $q_n = 2^n$, this proves (3.7). $\square$

3.3. **An obstruction for arbitrary bases.** Let $E = E(n,p)$ and let H be a finite-dimensional Hilbert space. For $z = (x,h) \in E \oplus_2 H$, we write $N(z) = N(x,h) = (\|x\|_E^2 + |h|_2^2)^{1/2}$ for its norm. When we consider the same vector space with the Hilbert norm $|z|_2 = (|x|_2^2 + |h|_2^2)^{1/2}$, we refer to this as its *coefficient Hilbert structure*, denoted by $\mathcal{H}_n \oplus_2 H$. Here $|x|_2$ is the Euclidean norm of the tensor coefficients of x. For a subspace $F \subseteq E \oplus_2 H$, let R_F be the orthogonal projection onto F in this Hilbert structure, and let $\iota_E\colon E \to E \oplus_2 H$ be the canonical inclusion. We define the *overlap* of F with E by

$$\theta(F) = q_n^{-1}\operatorname{tr}(\iota_E^* R_F \iota_E) \in [0,1]. \tag{3.9}$$

The quantity $\theta(F)$ measures the overlap of E and F in the coefficient Hilbert structure: it is the average squared length of the orthogonal projections onto F of the vectors in an orthonormal basis of E. It equals 1 precisely when $E \subseteq F$ and 0 precisely when $E \perp F$.

The next theorem bounds the overlap of F with E in terms of their relative dimensions and the unconditional basis constant of F, independently of the dimension of H. In the estimate below, we write $L_0 = (\gamma_p^2 + 1)^{1/2}$ and $\beta_p = 3^{(p-2)/(2p)}$.

Theorem 3.5. *Let $M \in \mathbb{N}$ and $C \geq 1$. Suppose $F \subseteq E \oplus_2 H$ has dimension at most Mq_n and a C-unconditional basis. Then*

$$\theta(F)\alpha_{p'}^{-n} \leq \beta_p C^{4-4/p} L_0^{(4-p)/p} M^{(p-2)/(2p)} \alpha_p^{4n(p-2)/p} \tau^{n(p-2)/(2p)} + C^2 L_0.$$

The bound is independent of the dimension of H.

Proof. We may assume $k = \dim F > 0$. We first compare the given basis with the coefficient Hilbert structure. After normalising the given C-unconditional basis $(b_i)_{i=1}^k$ of F so that $|b_i|_2 = 1$ for $i = 1,\dots,k$, let $(g_i)_{i=1}^k$ be its Hilbert biorthogonal family in F. Let $B\colon \ell_2^k \to F$ be given by $B((a_i)_{i=1}^k) = \sum_{i=1}^k a_i b_i$, and let D_ε be the basis multiplier with signs $\varepsilon = (\varepsilon_i)_{i=1}^k \in \{-1,1\}^k$. The inequalities $|z|_2 \leq N(z) \leq \alpha_p^n |z|_2$ imply $\|D_\varepsilon\|_{2\to 2} \leq C\alpha_p^n$. The normalisation of the basis gives, for $a = (a_i)_{i=1}^k \in \ell_2^k$,

$$\mathbb{E}_\varepsilon\big[|B((\varepsilon_i a_i)_{i=1}^k)|_2^2\big] = |a|_2^2.$$

Since each sign multiplier is its own inverse, the preceding bound applied in both directions yields

$$\|B\|_{2\to 2} \leq C\alpha_p^n, \qquad \|B^{-1}\|_{2\to 2} \leq C\alpha_p^n. \tag{3.10}$$

The corresponding operator for $(g_i)_{i=1}^k$ is $(B^{-1})^*$ and therefore satisfies the same bound.

Let x be a product unit vector as in Lemma 3.4, and put $z = R_F(x,0)$. By Lemma 3.1(ii), the functional represented by $(x,0)$ has dual norm at most $\alpha_{p'}^n$. Since R_F is an orthogonal projection, this gives

$$0 \leq \langle R_F(x,0),(x,0)\rangle = |z|_2^2 \leq \alpha_{p'}^n N(z).$$

Taking expectations and using $\mathbb{E}_x[xx^*] = I_{\mathcal{H}_n}/q_n$ to evaluate the quadratic form, we obtain

$$\mathbb{E}_x[N(z)] \geq \alpha_{p'}^{-n}\mathbb{E}_x\big[\langle R_F(x,0),(x,0)\rangle\big] = \frac{\alpha_{p'}^{-n}}{q_n}\operatorname{tr}(\iota_E^* R_F \iota_E) = \theta(F)\alpha_{p'}^{-n}. \tag{3.11}$$

To obtain an upper bound, we randomise the expansion of z in the given basis. For $i = 1,\dots,k$, write $b_i = (b_i^E, b_i^H)$ and $g_i = (g_i^E, g_i^H)$ according to the two summands, and define

$$S_E(x,t)^2 = \sum_{i=1}^k |\langle x, g_i^E\rangle|^2 |f_{b_i^E}(t)|^2,$$

$$V_H(x) = \sum_{i=1}^k |\langle x, g_i^E\rangle|^2 |b_i^H|_2^2.$$

Both expressions are quadratic in x, so their expectations are unchanged if x is replaced by a normalised Gaussian vector ζ. We obtain

$$\mathbb{E}[S_E^2] + \mathbb{E}[V_H] = \mathbb{E}_{\zeta,\varepsilon}[|D_\varepsilon R_F(\zeta,0)|_2^2] \le C^2 \mathbb{E}_\zeta[N(R_F(\zeta,0))^2],$$

where the equality follows by averaging over the independent signs and using $\mathbb{E}_t[|f_{b_i^E}(t)|^2] = |b_i^E|_2^2$ for $i = 1,\dots,k$. The inequality follows from $|D_\varepsilon u|_2 \le N(D_\varepsilon u) \le CN(u)$ for $u \in F$, by unconditionality.

The operator $\iota_E^* R_F \iota_E$ is a Hilbert contraction, so Lemma 3.1(iii) bounds the expected squared E-norm of the first component of $R_F(\zeta,0)$ by γ_p^2. The Hilbert component contributes at most $\mathbb{E}[|\zeta|_2^2] = 1$. Hence

$$\mathbb{E}[S_E^2] + \mathbb{E}[V_H] \le C^2 L_0^2. \tag{3.12}$$

Let $B_E, V_E \colon \ell_2^k \to \mathcal{H}_n$ be the operators with columns $(b_i^E)_{i=1}^k$ and $(g_i^E)_{i=1}^k$, respectively. These are the E-components of B and $(B^{-1})^*$. Since the coordinate projection onto E is contractive for the coefficient Hilbert norm, (3.10) gives $\|B_E\|, \|V_E\| \le C\alpha_p^n$. Applying Lemma 3.4 to these operators and the product unit vector x, and using $k/q_n \le M$, we obtain

$$\mathbb{E}_{x,t}[S_E^4] \le \frac{k}{q_n}(\|B_E\|\|V_E\|)^4\tau^n \le M(C\alpha_p^n)^8\tau^n = MC^8\alpha_p^{8n}\tau^n.$$

Since $p \in (2,3]$, interpolating between this bound and (3.12) yields

$$(\mathbb{E}[S_E^p])^{1/p} \le (C^2L_0^2)^{(4-p)/(2p)}\left(MC^8\alpha_p^{8n}\tau^n\right)^{(p-2)/(2p)}. \tag{3.13}$$

We observe that the Hilbertian distortion α_p^n appears in this bound only to the power $4(p-2)/p$, which tends to zero as $p \downarrow 2$. This permits the choice of parameters in Section 3.4, for which the distortion tends to infinity while its contribution $\alpha_p^{4n(p-2)/p}$ to the upper bound tends to one.

Let $(\varepsilon_i)_{i=1}^k$ be independent Rademacher variables. For scalars $(a_i)_{i=1}^k \in \mathbb{K}^k$, the sum $\sum_{i=1}^k \varepsilon_i a_i$ has second moment $\sum_{i=1}^k |a_i|^2$ and fourth moment at most $3(\sum_{i=1}^k |a_i|^2)^2$, as follows by expansion. Interpolating between these moments gives

$$\left(\mathbb{E}_\varepsilon\left[\left|\sum_{i=1}^k \varepsilon_i a_i\right|^p\right]\right)^{1/p} \le \beta_p\left(\sum_{i=1}^k |a_i|^2\right)^{1/2}.$$

Since $D_\varepsilon^{-1} = D_\varepsilon$, unconditionality implies $N(z) \le C\mathbb{E}_\varepsilon[N(D_\varepsilon z)]$. For fixed x, write $(u_\varepsilon, h_\varepsilon) = D_\varepsilon z = \sum_{i=1}^k \varepsilon_i\langle x, g_i^E\rangle b_i$. Then

$$\mathbb{E}_\varepsilon[N(D_\varepsilon z)] \le (\mathbb{E}_\varepsilon[\|u_\varepsilon\|_E^p])^{1/p} + (\mathbb{E}_\varepsilon[|h_\varepsilon|_2^2])^{1/2} \le \beta_p(\mathbb{E}_t[S_E(x,t)^p])^{1/p} + V_H(x)^{1/2},$$

where the first inequality follows from $N(u,h) \le \|u\|_E + |h|_2$ and Jensen's inequality. For the second, we apply the preceding scalar estimate pointwise in t and integrate, while $\mathbb{E}_\varepsilon[|h_\varepsilon|_2^2] = V_H(x)$ gives the Hilbert term. Averaging over x and using Jensen's inequality once more gives

$$\mathbb{E}_x[N(z)] \le C\left[\beta_p(\mathbb{E}_{x,t}[S_E^p])^{1/p} + (\mathbb{E}_x[V_H])^{1/2}\right]. \tag{3.14}$$

The estimates (3.12) and (3.13) now give the required upper bound. Comparing this with (3.11) proves the theorem. □

3.4. **Choosing the finite-dimensional parameters.** Using the dependence on α_p^n in (3.13), we choose the exponents so that the projection norms tend to one while the obstruction to unconditional bases becomes arbitrarily large. For $n \ge 1$, let

$$p_n = 2 + n^{-3/4}, \qquad E_n = E(n, p_n). \tag{3.15}$$

To isolate $\theta(F)$ in Theorem 3.5, we multiply by $\alpha_{p_n'}^n = c_{p_n}^n/\alpha_{p_n}^n$. The resulting bound contains the ratio $\tau^{n(p_n-2)/(2p_n)}/\alpha_{p_n}^n$, so we set

$$\Lambda_n = \alpha_{p_n}^n \tau^{-n(p_n-2)/(2p_n)}. \tag{3.16}$$

For fixed C and M, the remaining factors are bounded, as verified below, so Λ_n^{-1} controls the overlap. We have $\Lambda_n \le \alpha_{p_n}^n$; we shall see that both tend to infinity as $n \to \infty$.

In the real case, evaluating the L_r norm in (3.3) by the beta integral gives

$$\alpha_r^r = \frac{2^{r/2}\Gamma((r+1)/2)}{\sqrt{\pi}\Gamma(1+r/2)}.$$

Together with the explicit formula for α_r in the complex case, this yields

$$\frac{d}{dr}\log\alpha_r\Big|_{r=2} = \begin{cases}(1-\log 2)/4, & \mathbb{K}=\mathbb{R},\\ (\log 2 - 1/2)/4, & \mathbb{K}=\mathbb{C}.\end{cases}$$

Writing $\varepsilon = p-2$, we have $p' = 2-\varepsilon+O(\varepsilon^2)$, so the first-order terms cancel in $\log c_p$. For our choice $\varepsilon = p_n - 2 = n^{-3/4}$, the relations $n\varepsilon = n^{1/4} \to \infty$ and $n\varepsilon^2 = n^{-1/2} \to 0$ therefore give

$$c_{p_n}^n \longrightarrow 1, \qquad \alpha_{p_n}^n \longrightarrow \infty, \qquad \alpha_{p_n}^{4n(p_n-2)/p_n} \longrightarrow 1. \tag{3.17}$$

The same expansion applied to Λ_n gives

$$\log\Lambda_n = \kappa n^{1/4} + O(n^{-1/2}), \qquad \kappa = \begin{cases}(1-\log(5/2))/4, & \mathbb{K}=\mathbb{R},\\ (\log(9/5)-1/2)/4, & \mathbb{K}=\mathbb{C}.\end{cases} \tag{3.18}$$

Both values of κ are positive; for the complex value, use $e < 3 < (9/5)^2$.

We collect the Gaussian, trace and overlap estimates needed for the infinite sum. The last estimate shows that subspaces of dimension at most $3q_n$ with a uniformly bounded unconditional basis constant have overlap with E_n tending to zero, independently of the dimension of the adjoined Hilbert space.

Proposition 3.6. *For all sufficiently large n, independently of C and H, the following hold:*

(i) *For a normalised Gaussian vector ζ in $\mathcal{H}_n$,*

$$\mathbb{E}_\zeta[\|\zeta\|_{E_n}^2] \le 4.$$

(ii) *For every $T \in \mathcal{B}(E_n)$,*

$$\frac{|\operatorname{tr} T|}{q_n} \le \frac{2}{\alpha_{p_n}^n}\gamma_2(T).$$

(iii) *Let $C \ge 1$ and let H be a finite-dimensional Hilbert space. If $F \subseteq E_n \oplus_2 H$ has dimension at most $3q_n$ and a C-unconditional basis, then*

$$\theta(F) \le \frac{80C^3}{\Lambda_n}.$$

Proof. (i) and (ii) follow respectively from Lemma 3.1(iii) and Lemma 3.2, since $\gamma_{p_n} \to 1$.

For (iii), we apply Theorem 3.5 with $M = 3$ and multiply by $\alpha_{p_n'}^n = c_{p_n}^n/\alpha_{p_n}^n$. The fourth-moment factor divided by $\alpha_{p_n}^n$ is $1/\Lambda_n$, while the remaining scalar factors in both terms tend to $\sqrt{2}$ by (3.17). Since $4 - 4/p_n \le 3$, $C \ge 1$ and $\Lambda_n \le \alpha_{p_n}^n$, the estimate follows for all sufficiently large n, independently of C and H. □

Taking $F = E_n$ and $H = \{0\}$ gives $\theta(F) = 1$, hence $u(E_n) \ge (\Lambda_n/80)^{1/3} \to \infty$. This estimate concerns bases of the individual spaces. An unconditional basis of an infinite sum of these spaces need not have subsequences spanning the summands, so this estimate alone does not exclude such a basis. The overlap bound in Proposition 3.6(iii) also applies to subspaces of $E_n \oplus_2 H$ that need not contain E_n. The selection principle in Lemma 3.8 will select a small coordinate subspace from a proposed basis and, after replacing its complementary components by vectors in a Hilbert space, give a lower bound on its overlap with E_n. The recursive choice of summands then makes the two overlap bounds incompatible.

3.5. **An ℓ_2-sum without DPR-lust.** We now use these estimates to construct a space without DPR-lust. The main step is the finite selection principle of Lemma 3.8, which, starting from an unconditional basis, selects a finite-dimensional coordinate subspace and gives a lower bound for its overlap with a prescribed summand. Together with Proposition 3.6, this forces every unconditional basis of a finite-dimensional superspace of that summand to have a large constant.

3.5.1. *Local Hilbert approximation.* The next lemma shows that subspaces of fixed dimension in ℓ_2-sums of subspaces of L_r become uniformly close to Hilbert space as the exponents approach two. This remains true for subspaces of quotients of these sums.

Lemma 3.7. *For every integer $k \geq 1$ and $D > 1$ there is $\varepsilon(k, D) > 0$ with the following property. Let A be a nonempty index set. For each $\alpha \in A$, let $r_\alpha \in [2, 2 + \varepsilon(k, D)]$ and let X_α be a closed subspace of an L_{r_α} space, and set*

$$S = \left(\bigoplus_{\alpha \in A} X_\alpha \right)_2 .$$

Then every subspace of dimension at most k of every quotient of S is D-Hilbertian.

Proof. For $r \geq 2$ and $x, y \in L_r$, Clarkson's computation of the von Neumann–Jordan constant [7] gives

$$\nu_r^{-1}(\|x\|_r^2 + \|y\|_r^2) \leq \frac{\|x+y\|_r^2 + \|x-y\|_r^2}{2} \leq \nu_r(\|x\|_r^2 + \|y\|_r^2), \qquad \nu_r = 2^{1-2/r}. \tag{3.19}$$

Here the left inequality follows by applying the right one to $x + y$ and $x - y$. Summing squared norms shows that the same inequalities hold in S, with constant $\nu = \sup_{\alpha \in A} \nu_{r_\alpha}$.

To obtain the same inequalities in quotients of S, we use the *normalised Hadamard map* on $S \oplus_2 S$,

$$\mathsf{H}(x, y) = 2^{-1/2}(x + y, x - y),$$

which has norm at most $\sqrt{\nu}$ and preserves $N \oplus_2 N$ for every closed subspace $N \subseteq S$. It induces the same formula, with no larger norm, on $(S/N) \oplus_2 (S/N)$. The induced map squares to $I_{(S/N) \oplus_2 (S/N)}$, so both inequalities follow. They pass to subspaces as well, and their constants tend to one as all the exponents tend to two.

Passer's approximate Jordan–von Neumann theorem [26, Theorem 3.4] states that, in each fixed real dimension, the Banach–Mazur distance to Hilbert space tends to one as the constant in (3.19) tends to one. Over $\mathbb{R}$, applying this result in dimensions $2, \dots, k$, and observing that the one-dimensional case is automatic, proves the result. Over $\mathbb{C}$, we apply the same theorem in real dimensions at most $2k$. It gives a real Hilbert norm $|\cdot|_0$ satisfying $|x|_0 \leq \|x\| \leq D|x|_0$. The averaged norm

$$|x|_H^2 = \frac{|x|_0^2 + |ix|_0^2}{2}$$

satisfies the same comparison and is invariant under multiplication by i, so it comes from a complex inner product. □

Before we move on, we observe that the lemma applies in particular to subspaces of S itself of dimension at most k. Moreover, the choice of $\varepsilon(k, D)$ may depend on the scalar field.

3.5.2. *A finite selection principle.* For the selection lemma, let E be a q-dimensional Banach space with a specified Hilbert norm $|\cdot|_2$ satisfying $|x|_2 \leq \|x\|_E$. We use the same notion of overlap as in Section 3.3: if H is a finite-dimensional Hilbert space and $F \subseteq E \oplus_2 H$, we write

$$\theta_E(F) = \frac{1}{q} \operatorname{tr}(\iota_E^* R_F \iota_E),$$

where R_F is the orthogonal projection in the coefficient Hilbert structure and ι_E is the inclusion of the first summand.

The next lemma is the key step in passing from the finite-dimensional overlap estimate to failure of DPR-lust. Under its hypotheses, it selects at most $3q$ vectors from an arbitrary unconditional basis and, after a Hilbertian replacement of their complementary components, produces a subspace of $E \oplus_2 H$ with a controlled unconditional basis constant and a quantitative lower bound on its overlap with E. In Theorem 3.9, comparison with the upper bound in Proposition 3.6(iii) forces the unconditional basis constants of all finite-dimensional superspaces of the chosen summands to grow. This is what allows us to rule out DPR-lust even though these bases need not respect the given decomposition.

Lemma 3.8. *Suppose that $Y = E \oplus_2 W$ has a C-unconditional basis and that every subspace of W of dimension at most $3q$ is D-Hilbertian. Suppose that there are constants $a, L > 0$ such that*

$$\frac{|\operatorname{tr} T|}{q} \le a\gamma_2(T) \quad (T \in \mathcal{B}(E)), \qquad \mathbb{E}[\|\zeta\|_E^2] \le L^2, \tag{3.20}$$

where ζ is the normalised Gaussian vector in E. If $\eta = aDC^2 \le 1/8$, there are a finite-dimensional Hilbert space H and a subspace $F' \subseteq E \oplus_2 H$ such that

(i) $\dim F' \le 3q$.

(ii) *F' has a DC-unconditional basis.*

(iii) $\theta_E(F') \ge \dfrac{(1-4\eta)^2}{C^2L^2}$.

Proof. Let $(z_i, z_i^*)_{i\in I}$ be the basis and its coordinate functionals, where $I = \mathbb{N} = \{1, 2, \dots\}$ in the infinite-dimensional case and $I = \{1, \dots, m\}$ if $\dim Y = m < \infty$. Let $\iota_E\colon E \to Y$ be the canonical inclusion, and let $P_E\colon Y \to E$ and $P_W\colon Y \to W$ be the coordinate projections associated with $Y = E \oplus_2 W$. Put

$$Q_i = z_i \otimes z_i^*, \qquad A_i = P_E Q_i \iota_E, \qquad t_i = \operatorname{tr} A_i \quad (i \in I).$$

Each A_i has rank at most one, so $\operatorname{tr}(A_i^2) = t_i^2$. The partial sums of the Q_i converge to the identity uniformly on the unit ball of the finite-dimensional space $\iota_E(E)$. Thus $\sum_{i\in I} t_i = q$.

For a finite set $J \subseteq I$, scalars $(w_i)_{i\in J}$ with $|w_i| \le 1$ for $i \in J$, and independent Rademacher signs $(\varepsilon_i)_{i\in J}$, put

$$M_\varepsilon = \sum_{i\in J} \varepsilon_i Q_i, \qquad N_\varepsilon = \sum_{i\in J} \varepsilon_i w_i Q_i.$$

Both operators have norm at most C. Write $T_\varepsilon, S_\varepsilon$ for their compressions to E, and $B_\varepsilon, V_\varepsilon$ for the upper-right block of M_ε and the lower-left block of N_ε, respectively. Since $Q_iQ_k = \delta_{ik}Q_i$,

$$T_\varepsilon S_\varepsilon + B_\varepsilon V_\varepsilon = \sum_{i\in J} w_i A_i.$$

The range of V_ε has dimension at most q and is D-Hilbertian, so $\gamma_2(B_\varepsilon V_\varepsilon) \le DC^2$. Taking traces and averaging over the signs therefore gives

$$\left|\sum_{i\in J} w_i(t_i - t_i^2)\right| = |\mathbb{E}[\operatorname{tr}(B_\varepsilon V_\varepsilon)]| \le \eta q.$$

Choosing w_i so that $w_i(t_i - t_i^2) = |t_i - t_i^2|$ and passing to increasing finite sets yields, in particular,

$$\sum_{i\in I} |t_i - t_i^2| \le 2\eta q. \tag{3.21}$$

Set

$$S = \{i \in I : |t_i - 1| \le 1/2\}, \qquad Q_S = \sum_{i\in S} Q_i, \qquad F = Q_S Y.$$

The set S is finite, since $t_i \to 0$ when $I = \mathbb{N}$. Outside S, the inequality $|t_i| \le 2|t_i - t_i^2|$ holds. Hence

$$(1-4\eta)q \le \operatorname{Re}\sum_{i\in S} t_i \le (1+4\eta)q. \tag{3.22}$$

For every $i \in S$, we have $\operatorname{Re} t_i \geq 1/2$, so $|S| \leq 2(1+4\eta)q \leq 3q$. Moreover, $\|Q_S\| \leq C$, and the selected vectors $(z_i)_{i\in S}$ form a C-unconditional basis of F.

It remains to replace the W-component of F by a Hilbert space. Since $\dim P_W F \leq 3q$, there is an isomorphism $J\colon P_W F \to H$ onto a Hilbert space such that $\|J\| \leq 1$ and $\|J^{-1}\| \leq D$. Define $U\colon E \oplus_2 P_W F \to E \oplus_2 H$ by $U(x,w) = (x, Jw)$ and let $F' = UF$. This space has the same dimension as F, which proves (i). The image under U of the selected basis is DC-unconditional, proving (ii). To prove (iii), define

$$T_0 = UQ_S\iota_E\colon E \longrightarrow F'.$$

Write ι_E also for the inclusion into $E \oplus_2 H$. The norm of $T_0\colon E \to F'$ is at most C, and U fixes the E component, so (3.22) yields

$$\operatorname{Re}\operatorname{tr}(\iota_E^* T_0) \geq (1-4\eta)q.$$

The coefficient Hilbert norm on $E \oplus_2 H$ is bounded above by its Banach norm. Therefore

$$\|T_0\|_{\mathrm{HS}}^2 = q\,\mathbb{E}[|T_0\zeta|_2^2] \leq qC^2L^2.$$

Let R be the orthogonal projection onto F'. As $RT_0 = T_0$, Cauchy–Schwarz for the Hilbert–Schmidt inner product gives

$$(1-4\eta)q \leq \operatorname{Re}\langle T_0, R\iota_E\rangle_{\mathrm{HS}} \leq \|T_0\|_{\mathrm{HS}}\|R\iota_E\|_{\mathrm{HS}} \leq CLq\,\theta_E(F')^{1/2}.$$

This proves (iii) and completes the proof. □

3.5.3. *Choosing the summands.* We return to the spaces $E_n \subseteq L_{p_n}$ and the constants Λ_n from Proposition 3.6, with $q_n = \dim E_n$. Recall that

$$p_n \downarrow 2, \qquad \Lambda_n = \alpha_{p_n}^n \tau^{-n(p_n-2)/(2p_n)} \longrightarrow \infty, \qquad \Lambda_n \leq \alpha_{p_n}^n.$$

We shall recursively choose a strictly increasing sequence $(n_j)_{j=1}^\infty$ of integers, all sufficiently large for the conclusions of Proposition 3.6 to hold. To simplify notation, write

$$E_j = E_{n_j}, \quad p_j = p_{n_j}, \quad q_j = q_{n_j}, \quad \Lambda_j = \Lambda_{n_j} \qquad (j \geq 1),$$

and put $D_1 = 2$ and $D_j = \max\{2, \alpha_{p_{j-1}}^{n_{j-1}}\}$ for $j \geq 2$. At each step, we also arrange that

$$\Lambda_j \geq D_j^8, \qquad p_j - 2 \leq \varepsilon(3q_i, 2) \quad (1 \leq i < j). \tag{3.23}$$

Once $n_1, \dots, n_{j-1}$ have been chosen, D_j and the finitely many values $\varepsilon(3q_i, 2)$ for $1 \leq i < j$ are fixed. Since $p_n \to 2$ and $\Lambda_n \to \infty$, these conditions can be satisfied by taking n_j sufficiently large. These choices also make $(\alpha_{p_j}^{n_j})_{j=1}^\infty$ increasing, because $\alpha_{p_j}^{n_j} \geq \Lambda_j \geq D_j^8$.

Set

$$Y = \left(\bigoplus_{j=1}^\infty E_j\right)_2, \qquad W_j = \left(\bigoplus_{\substack{i=1\\ i\neq j}}^\infty E_i\right)_2 \qquad (j \geq 1). \tag{3.24}$$

Every subspace $V \subseteq W_j$ of dimension at most $3q_j$ is D_j-Hilbertian. Indeed, its projection onto the later summands is 2-Hilbertian by Lemma 3.7 and (3.23). For $v \in V$, write $v = v_- + v_+$, where v_- and v_+ are its coordinate projections onto $(\bigoplus_{i=1}^{j-1} E_i)_2$ and $(\bigoplus_{i=j+1}^\infty E_i)_2$, respectively. Writing $v_- = (v_i)_{i=1}^{j-1}$, we use the coefficient Hilbert norms of Section 2.2 to define $|v_-|_2^2 = \sum_{i=1}^{j-1} |v_i|_2^2$. By Lemma 3.1,

$$|v_-|_2 \leq \|v_-\| \leq D_j |v_-|_2.$$

Choose a Hilbert norm $|\cdot|_+$ on the later projection of V such that $|v_+|_+ \leq \|v_+\| \leq 2|v_+|_+$, and define a Hilbert norm on V by

$$|v|_H^2 = |v_-|_2^2 + |v_+|_+^2.$$

Since $D_j \geq 2$, this norm satisfies $|v|_H \leq \|v\| \leq D_j |v|_H$ for every $v \in V$.

The next theorem compares the lower overlap bound from Lemma 3.8(iii) with the upper bound in Proposition 3.6(iii). The recursive choice of summands forces unconditional basis constants to grow with j in every closed subspace of Y containing E_j, thereby ruling out both DPR-lust and an unconditional basis of Y.

Theorem 3.9. *Let Y be the space in (3.24). For every $j \geq 1$, if F is a closed subspace with $E_j \subseteq F \subseteq Y$, then every unconditional basis of F has constant greater than $\Lambda_j^{1/8}/5$. In particular,*

$$\inf\{u(F) : E_j \subseteq F \subseteq Y,\ \dim F < \infty\} \geq \frac{\Lambda_j^{1/8}}{5} \longrightarrow \infty \qquad (j \to \infty).$$

Consequently, Y has neither DPR-lust nor an unconditional basis.

Proof. Fix $j \geq 1$ and such a subspace F. The coordinate projection onto E_j leaves F invariant, because $E_j \subseteq F$. Hence

$$F = E_j \oplus_2 (F \cap W_j)$$

isometrically. Every subspace of $F \cap W_j$ of dimension at most $3q_j$ is D_j-Hilbertian by the preceding estimates for the earlier and later summands. Suppose that F has a C-unconditional basis with $C \leq \Lambda_j^{1/8}/5$. We apply Lemma 3.8 with $a_j = 2/\alpha_{p_j}^{n_j}$ and $L_j \leq 2$, by Proposition 3.6. Since $D_j \leq \Lambda_j^{1/8}$, $\alpha_{p_j}^{n_j} \geq \Lambda_j \geq 1$ and

$$\eta_j = \frac{2D_jC^2}{\alpha_{p_j}^{n_j}} \leq \frac{2}{25}\Lambda_j^{-5/8} < \frac{1}{8},$$

the lemma gives a subspace $F' \subseteq E_j \oplus_2 H$ of dimension at most $3q_j$, with a D_jC-unconditional basis and $\theta_{E_j}(F') \geq 1/(16C^2)$. The finite-dimensional overlap estimate therefore yields

$$\frac{1}{16C^2} \leq \theta_{E_j}(F') \leq \frac{80(D_jC)^3}{\Lambda_j} \leq 80C^3\Lambda_j^{-5/8}.$$

It follows that

$$1 \leq 1280C^5\Lambda_j^{-5/8} \leq \frac{1280}{5^5} < 1,$$

a contradiction. The argument applies to every closed subspace F containing E_j, since the Hilbert approximation is needed only on the subspace selected by the lemma. Restricting to finite-dimensional F proves failure of DPR-lust, while taking $F = Y$ for all $j \geq 1$ rules out an unconditional basis. ☐

Growth of $u(E_j)$ alone would not exclude DPR-lust, whose definition allows passage to a larger finite-dimensional subspace of Y. We next realise Y as the range of a projection with norm arbitrarily close to one on a space with a 1-unconditional basis.

3.6. **Finite-dimensional embeddings.** We now embed the tensor spaces into finite-dimensional ℓ_p spaces. Taking the ℓ_2-sum of these embeddings will realise an isomorphic copy of Y as a complemented subspace of X_ρ, with a projection of norm less than $1+\rho$. Over $\mathbb{R}$, a uniform circle grid preserves covariance exactly and gives explicit projections, whereas over $\mathbb{C}$, conditional expectations onto finite partitions give the required approximation.

3.6.1. *The real case.* For $\mathbb{K} = \mathbb{R}$, we sample the functions f_x on a uniform grid in $\mathbb{T}^n$.

For each $n \geq 1$, put $M = 1000n^2$ and $\vartheta_r = 2\pi r/M$ for $r \in \{0, \dots, M-1\}$. Let U_n be the space of real functions on $\{0, \dots, M-1\}^n$, with the L_{p_n} norm for normalised counting measure. Define operators $A_n \colon E_n \to U_n$ and $B_n \colon U_n \to E_n$ by

$$\begin{aligned}(A_nx)_r &= f_x(\vartheta_{r_1}, \dots, \vartheta_{r_n}),\\ B_nz &= M^{-n} \sum_{r\in\{0,\dots,M-1\}^n} z_r W(\vartheta_{r_1}, \dots, \vartheta_{r_n}).\end{aligned} \tag{3.25}$$

For $M \geq 3$, the grid averages of $\cos(2t)$ and $\sin(2t)$ vanish, so $M^{-1}\sum_{r=0}^{M-1} w(\vartheta_r)w(\vartheta_r)^{\mathsf{T}} = I_{\mathbb{R}^2}$. Taking tensor products therefore gives

$$B_nA_n = I_{E_n}. \tag{3.26}$$

Thus A_nB_n is a projection onto $A_n(E_n)$. The next lemma estimates these maps and shows that the projection norms tend to one.

Lemma 3.10. *With $b_M = 1 + 64\pi/M$ and $p = p_n$, we have*

$$\|A_n\| \le b_M^n, \qquad \|B_n\| \le (c_p b_M)^n, \qquad \|A_nB_n\| \le (c_p b_M^2)^n \longrightarrow 1. \tag{3.27}$$

Proof. For $1 \le r \le 4$, the function $|\sqrt{2}\cos(t-\phi)|^r$ is 16-Lipschitz, uniformly in ϕ, so its grid average differs from its integral by at most $32\pi/M$. By (3.3) and translation invariance, its integral equals α_r^r. This integral is at least $1/2$: for $r \ge 2$, monotonicity of the L_r norms on the normalised circle gives $\alpha_r^r \ge \alpha_2^r = 1$, while for $1 \le r \le 2$ we have $\alpha_1 = 2\sqrt{2}/\pi < 1$ and

$$\alpha_r^r \ge \alpha_1^r \ge \alpha_1^2 = \frac{8}{\pi^2} > \frac{1}{2}.$$

The preceding estimates show that every real linear combination of $\cos t$ and $\sin t$ has discrete L_r norm at most b_M times its $L_r(\mathbb{T})$ norm.

For $x \in E_n$, fixing all but one variable makes f_x a real linear combination of $\cos t$ and $\sin t$ in that variable. Apply the comparison at exponent p successively in each variable, integrating or averaging over the remaining variables at each step. This gives

$$\|A_n x\|_{U_n} \le b_M^n \|f_x\|_{L_p(\mathbb{T}^n)} = b_M^n \|x\|_{E_n}.$$

To estimate B_n, first consider $z = (z_r)_{r=0}^{M-1}$ on the one-dimensional grid and put $v = M^{-1}\sum_{r=0}^{M-1} z_r w(\vartheta_r)$. For every $a \in \mathbb{R}^2$ with $|a|_2 = 1$, we have

$$|\langle v, a\rangle| \le \left(M^{-1}\sum_{r=0}^{M-1}|z_r|^p\right)^{1/p}\left(M^{-1}\sum_{r=0}^{M-1}|\langle w(\vartheta_r), a\rangle|^{p'}\right)^{1/p'} \le b_M\alpha_{p'}\left(M^{-1}\sum_{r=0}^{M-1}|z_r|^p\right)^{1/p},$$

where the first inequality is Hölder's and the second follows from the grid comparison at exponent p' and $\|\langle w(\cdot), a\rangle\|_{L_{p'}(\mathbb{T})} = \alpha_{p'}$. Taking the supremum over a and recalling $c_p = \alpha_p\alpha_{p'}$, we obtain

$$\|\langle v, w(\cdot)\rangle\|_{L_p(\mathbb{T})} = \alpha_p|v|_2 \le c_p b_M \left(M^{-1}\sum_{r=0}^{M-1}|z_r|^p\right)^{1/p}.$$

For $z \in U_n$, apply this map in each coordinate, keeping the other coordinates fixed. At each step, integrate the p-th power of the one-variable estimate over the remaining coordinates. Fubini's theorem then gives a factor $c_p b_M$ in the product L_p norm at each step. After all n coordinates have been treated, the resulting function is $f_{B_n z}$ by (3.25), so

$$\|B_n z\|_{E_n} = \|f_{B_n z}\|_{L_p(\mathbb{T}^n)} \le (c_p b_M)^n \|z\|_{U_n}.$$

Finally, $\|A_nB_n\| \le \|A_n\|\|B_n\| \le (c_p b_M^2)^n$. By (3.17), $n\log c_{p_n} \to 0$, while $M = 1000n^2$ gives

$$0 \le n\log b_M \le \frac{64\pi n}{M} = \frac{64\pi}{1000n} \longrightarrow 0.$$

The upper bound for the projection norm therefore tends to one. ☐

3.6.2. *The complex case.* The complex analogue uses the continuous projection P_n from Section 3.1. The next lemma obtains the finite-dimensional embedding by conditional expectations; see [16, Section 5.1, Theorem 33] for similar quantitative results on complemented embeddings into finite-dimensional ℓ_p spaces.

Lemma 3.11. *Suppose $\mathbb{K} = \mathbb{C}$. For each $n \ge 1$ and $0 < \delta < 1$, there are a finite-dimensional complex ℓ_{p_n} space U and operators $A\colon E_n \to U$ and $B\colon U \to E_n$ such that*

$$BA = I_{E_n}, \qquad \|A\| \le 1, \qquad \|B\| \le \frac{c_{p_n}^n}{1-\delta}.$$

Proof. Let $(C_r)_{r=1}^{\infty}$ be conditional expectations onto increasing finite partitions generating the product sphere σ-algebra. Their strong convergence on L_{p_n} and the finite dimension of E_n imply that $T_r = P_n C_r|_{E_n} \to I_{E_n}$ in operator norm. Choose r so that $\|I_{E_n} - T_r\| < \delta$ and set

$$U = \operatorname{ran} C_r, \qquad A = C_r|_{E_n}, \qquad B = T_r^{-1} P_n|_U.$$

Then $BA = I_{E_n}$, and the bounds for A and B follow from $\|C_r\| = 1$, $\|P_n\| \le c_{p_n}^n$ and $\|T_r^{-1}\| \le (1-\delta)^{-1}$. The norm on U is a weighted ℓ_{p_n} norm, so rescaling the coordinates identifies U isometrically with $\ell_{p_n}^N$, where $N = \dim U$. □

3.6.3. *The complemented embedding.* For a uniformly bounded family of operators $(T_j \colon V_j \to W_j)_{j\in J}$ between Banach spaces, we denote by $\operatorname{diag}(T_j : j \in J)$ the *diagonal operator* from $(\bigoplus_{j\in J} V_j)_2$ to $(\bigoplus_{j\in J} W_j)_2$ given by

$$\operatorname{diag}(T_j : j \in J)\big((x_j)_{j\in J}\big) = (T_j x_j)_{j\in J}.$$

Its norm is $\sup_{j\in J}\|T_j\|$. We now combine the embeddings over either field.

Construction of the projection. By Lemmas 3.10 and 3.11 and (3.17), the sequence $(n_j)_{j=1}^{\infty}$ in Theorem 3.9 can be chosen so that there are $U_j = \ell_{p_j}^{N_j}$ and uniformly bounded maps $A_j \colon E_j \to U_j$ and $B_j \colon U_j \to E_j$ satisfying

$$B_j A_j = I_{E_j}, \qquad \|A_j B_j\| \le 1 + \rho/2 \qquad (j \ge 1).$$

Over $\mathbb{R}$, these are the sampling maps A_{n_j}, B_{n_j} after rescaling the coordinates of U_{n_j}. Over $\mathbb{C}$, choose errors $(\delta_j)_{j=1}^{\infty}$ tending to zero so that $c_{p_j}^{n_j}/(1-\delta_j) \le 1 + \rho/2$. For $X = (\bigoplus_{j=1}^{\infty} U_j)_2$, the diagonal maps

$$A = \operatorname{diag}(A_j : j \in \mathbb{N}) \colon Y \to X, \qquad B = \operatorname{diag}(B_j : j \in \mathbb{N}) \colon X \to Y$$

are bounded and satisfy $BA = I_Y$. Thus A is an isomorphism onto the closed subspace $Z = A(Y)$, and $P = AB$ is a projection onto Z. Its norm is the supremum of the block norms, so $\|P\| \le 1 + \rho/2 < 1 + \rho$.

Write $P_j = A_j B_j$ for $j \ge 1$. In the real case, with $M_j = 1000 n_j^2$ and $N_j = M_j^{n_j}$, the inner product of two sampling vectors gives the explicit matrix formula

$$P_j(r,s) = \left(\frac{2}{M_j}\right)^{n_j} \prod_{t=1}^{n_j} \cos\left(\frac{2\pi(r_t - s_t)}{M_j}\right), \qquad P = \operatorname{diag}(P_j : j \in \mathbb{N}), \tag{3.28}$$

where $r, s \in \{0, \dots, M_j - 1\}^{n_j}$. In this case, (3.26) shows that $P_j^2 = P_j$ and $\operatorname{rank} P_j = 2^{n_j}$; symmetry makes P_j a Hilbert orthogonal projection. Multiplying the coordinates of U_{n_j} by $M_j^{-n_j/p_j}$ identifies this space isometrically with $\ell_{p_j}^{N_j}$ and leaves the matrix of P_j unchanged.

Over either field, taking the coordinate bases in block order gives a 1-unconditional basis of X. The space Z is isomorphic to Y and therefore fails DPR-lust by Theorem 3.9. The ℓ_2-sum representation also shows that X and Z are separable and reflexive. □

To complete the proofs of Theorems A and B, we must still establish the GL-lust bounds, superreflexivity and the conclusions for the dual and, over $\mathbb{R}$, both complementary summands. We do so in Section 4, where we also obtain the Banach-lattice conclusions.

4. Local unconditional structure and Banach lattices

We combine the failure of DPR-lust in Theorem 3.9 with the complemented embedding of Section 3.6. The first two subsections apply over either scalar field and establish the separation for $Z = P(X)$ and its dual. In the real case, we then modify the construction to obtain the same conclusions for both complementary summands, completing Theorem A.

4.1. **Separation of GL-lust and DPR-lust.** We begin with the standard estimate for GL-lust on complemented subspaces.

Lemma 4.1. *Let X be a Banach space with a 1-unconditional basis, and let Z be a nonzero Banach space. If bounded linear operators $J\colon Z \to X$ and $R\colon X \to Z$ satisfy $RJ = I_Z$, then*

$$1 \le \chi_{\mathrm{GL}}(Z) \le \|J\|\|R\|.$$

Proof. Since X has a 1-unconditional basis, approximation by finite coordinate spans gives $\chi_{\mathrm{GL}}(X) = 1$. The factorisation argument of Gordon and Lewis [14, Lemma 3.2], applied to $J|_V$ and R for each finite-dimensional subspace $V \subseteq Z$, gives

$$\chi_{\mathrm{GL}}(Z) \le \|J\|\|R\|\,\chi_{\mathrm{GL}}(X) = \|J\|\|R\|.$$

The lower bound follows because every inclusion of a nonzero subspace has norm one. □

Combining Lemma 4.1 with Theorem 3.9 gives the GL/DPR separation for $Z = P(X)$. We also record the geometric properties that follow from its ℓ_2-sum structure.

Corollary 4.2. *Let P be the projection constructed in Section 3.6, and set $Z = P(X)$. Then:*

(i) *The local unconditional constants satisfy*

$$\chi_{\mathrm{GL}}(Z) \le \|P\| < 1 + \rho, \qquad \chi_{\mathrm{DPR}}(Z) = \infty.$$

Consequently, Z admits no unconditional basis. If $\mathbb{K} = \mathbb{R}$, it is not isomorphic to any Banach lattice.

(ii) *Both X and Z are uniformly convex and hence superreflexive.*

(iii) *Z has a contractive unconditional finite-dimensional decomposition and the metric approximation property.*

Proof. For (i), observe that Z is isomorphic to the space Y of Theorem 3.9, and therefore fails DPR-lust and admits no unconditional basis. Moreover, Lemma 4.1 gives $\chi_{\mathrm{GL}}(Z) \le \|P\|$. If $\mathbb{K} = \mathbb{R}$, the nonlattice conclusion follows because every Banach lattice has DPR-lust and this property is invariant under isomorphism.

For (ii), recall that $2 < p_j \le 3$ for every $j \ge 1$. Clarkson's inequality [6, Theorem 2] gives the following common lower bound for the moduli of the summands:

$$\delta_{\ell_p^N}(\varepsilon) \ge 1 - \big(1 - (\varepsilon/2)^p\big)^{1/p} \ge \frac{\varepsilon^3}{24} \quad (2 \le p \le 3,\ N \ge 1,\ 0 < \varepsilon \le 2).$$

An ℓ_2-sum of spaces with a common modulus of uniform convexity is uniformly convex, so both X and its subspace Z have this property.

To prove (iii), let $X_j = \ell_{p_j}^{N_j}$ and $Z_j = P_j X_j$ for $j \ge 1$. Since P acts diagonally, we have the isometric decomposition

$$Z = \left(\bigoplus_{j=1}^{\infty} Z_j\right)_2.$$

Its coordinate decomposition is contractive and unconditional, so Z has the metric approximation property in its inherited norm, as observed in Section 2.4. □

4.2. **Duality.** We next show that Y^* also fails DPR-lust. The proof uses the local Hilbert estimate for quotients in Lemma 3.7, which allows us to work with the same sequence of summands as before. We use the canonical isometric identification $Y^* = E_j^* \oplus_2 W_j^*$ for $j \ge 1$.

Proposition 4.3. *For the space Y in Theorem 3.9,*

$$\lambda_{\mathrm{DPR}}(Y^*; E_j^*) \geq \frac{\Lambda_j^{1/8}}{10} \longrightarrow \infty \qquad (j \to \infty).$$

In particular, Y^* fails DPR-lust.

Proof. Let $j \geq 1$ and let F be a finite-dimensional subspace satisfying $E_j^* \subseteq F \subseteq Y^*$. Since F contains the entire coordinate summand E_j^*, we have $F = E_j^* \oplus_2 V$ isometrically, where $V = F \cap W_j^*$. Consequently, $F^* = E_j \oplus_2 V^*$. By reflexivity and Hahn–Banach, V^* is isometric to the quotient $W_j/V_\perp$, where $V_\perp = \{w \in W_j : f(w) = 0 \text{ for every } f \in V\}$.

To apply the selection lemma to F^*, we need a local Hilbert estimate for V^*. Let S_j be the space obtained from W_j by equipping its earlier summands with their coefficient Hilbert norms and leaving the later summands unchanged. The identity from S_j to W_j induces an isomorphism

$$S_j/V_\perp \longrightarrow W_j/V_\perp = V^*$$

with distortion at most D_j. The sum of the earlier summands in S_j is Hilbertian, while $p_i - 2 \leq \varepsilon(3q_j, 2)$ for every $i > j$. Combining Lemma 3.7 with this distortion bound shows that every subspace of V^* of dimension at most $3q_j$ is $2D_j$-Hilbertian.

Suppose now that F has a C-unconditional basis with $C \leq \Lambda_j^{1/8}/10$. The dual basis of F^* is also C-unconditional. We may apply Lemma 3.8 to F^* with $a = 2/\alpha_{p_j}^{n_j}$, $L \leq 2$ and $D = 2D_j$, since

$$\eta = \frac{4D_jC^2}{\alpha_{p_j}^{n_j}} \leq \frac{1}{25}\Lambda_j^{-5/8} < \frac{1}{8}.$$

The resulting subspace $F' \subseteq E_j \oplus_2 H$ satisfies the lower estimate in Lemma 3.8(iii) and the upper estimate in Proposition 3.6(iii):

$$\frac{1}{16C^2} \leq \theta_{E_j}(F') \leq \frac{80(2D_jC)^3}{\Lambda_j} \leq 640C^3\Lambda_j^{-5/8}.$$

Consequently $1 \leq 10240C^5\Lambda_j^{-5/8} \leq 10240/10^5 < 1$, a contradiction. □

Together with the duality of GL-lust, Proposition 4.3 gives the analogue of Corollary 4.2 for Z^*.

Corollary 4.4. *For $Z = P(X)$ as in Corollary 4.2, the following hold:*

(i) *The local unconditional constants of Z^* satisfy*

$$\chi_{\mathrm{GL}}(Z^*) \leq \|P\| < 1 + \rho, \qquad \chi_{\mathrm{DPR}}(Z^*) = \infty.$$

Consequently, Z^* admits no unconditional basis. If $\mathbb{K} = \mathbb{R}$, it is not isomorphic to any Banach lattice.

(ii) *Z^* is superreflexive.*

(iii) *Z^* has a contractive unconditional finite-dimensional decomposition and the metric approximation property.*

Proof. For (i), the duality identity for GL-lust [28, Remark 1.1] and Corollary 4.2(i) give

$$\chi_{\mathrm{GL}}(Z^*) = \chi_{\mathrm{GL}}(Z) \leq \|P\|.$$

The isomorphism between Z^* and Y^*, together with Proposition 4.3, shows that Z^* fails DPR-lust. It therefore admits no unconditional basis and, if $\mathbb{K} = \mathbb{R}$, cannot be isomorphic to a Banach lattice.

(ii) follows from Corollary 4.2(ii), since superreflexivity passes to duals. For (iii), use the isometric block decomposition in the proof of Corollary 4.2(iii). Writing $Z_j = P_j(\ell_{p_j}^{N_j})$ for $j \geq 1$, we obtain $Z^* = (\bigoplus_{j=1}^\infty Z_j^*)_2$ isometrically. This decomposition is contractive and unconditional, and its initial block projections give the metric approximation property. □

4.3. **Two complementary summands without lattice structures.** For the remainder of this section, we work over the real field. To obtain the separation for both complementary summands, we shall alternate the finite projections with their complements. This requires a stronger recursive choice, controlling the Hilbertian distortion of all preceding coordinate spaces. Stern's estimate for projections on L_p spaces [31] will ensure that the complementary projections also have norms tending to one.

Recall that a class of Banach spaces, closed under isomorphism, is called *primary* if, whenever X belongs to the class and $X = Y \oplus Z$ is a topological decomposition, at least one of Y and Z also belongs to the class. For the class of Banach lattices, membership is understood up to Banach-space isomorphism.

The next corollary gives the GL/DPR separation for both complementary summands and their duals, proving Corollary C.

Corollary 4.5. *For every $\rho > 0$ there are a separable superreflexive real Banach lattice X with a 1-unconditional basis and a projection Q on X such that*

$$\|Q\| < 1 + \rho, \qquad \|I_X - Q\| < 1 + \rho.$$

The spaces QX, $(I_X - Q)X$ and their duals satisfy:

(i) *Each has GL-lust but fails DPR-lust, and none is isomorphic to a Banach lattice.*

(ii) *Each has a contractive unconditional finite-dimensional decomposition and the metric approximation property.*

Consequently, the class of separable real Banach lattices is not primary.

Proof. For each $n \geq 1$, let $R_n = A_n B_n$ be the projection on the sampling space U_n from Section 3.6. We have $\dim U_n = (1000n^2)^n$ and $\operatorname{rank} R_n = 2^n$. Thus both R_n and $I_{U_n} - R_n$ are nonzero. Since $U_n = \ell_{p_n}^{(1000n^2)^n}$ with its normalised counting measure, Stern's projection estimate, together with his computation of the Banach–Mazur constant of L_{p_n} spaces [31, Sections 4 and 5], gives

$$1 \leq \|I_{U_n} - R_n\| \leq 2^{1-2/p_n} \|R_n\| \longrightarrow 1,$$

where the convergence follows from $p_n \to 2$ and Lemma 3.10. We may therefore restrict to indices for which $\max\{\|R_n\|, \|I_{U_n} - R_n\|\} \leq 1 + \rho/2$ and $\max\{\|A_n\|, \|B_n\|\} \leq 2$.

We now choose $(n_j)_{j=1}^\infty$ as in Section 3.5, with one modification. We write $E_j = E_{n_j}$, $V_j = U_{n_j}$ and $N_j = \dim V_j$, and retain the notation p_j, q_j, Λ_j for the other selected quantities, for $j \geq 1$. The Hilbert bound at the jth stage is taken to be

$$D_j = \max\Big(\{2\} \cup \{\alpha_{p_i}^{n_i}, N_i^{1/2-1/p_i} : 1 \leq i < j\}\Big),$$

and we retain the conditions

$$\Lambda_j \geq D_j^8, \qquad p_j - 2 \leq \varepsilon(3q_i, 2) \quad (1 \leq i < j). \tag{4.1}$$

Since D_j depends only on the preceding choices, the recursion remains possible. The additional term accounts for the entire space V_i, whose Hilbert distortion is at most $N_i^{1/2-1/p_i}$ by the usual comparison of finite-dimensional ℓ_p norms.

We claim that, with this choice, every sum of the form

$$Z = \left(\bigoplus_{i=1}^\infty Z_i \right)_2, \qquad Z_i = E_i \text{ or } Z_i \subseteq V_i \quad (i \geq 1)$$

fails DPR-lust whenever $J = \{j \in \mathbb{N} : Z_j = E_j\}$ is infinite. To see this, fix $j \in J$ and consider a subspace of dimension at most $3q_j$ of $W_j = (\bigoplus_{i=1, i \neq j}^\infty Z_i)_2$. Its projection onto the earlier summands is D_j-Hilbertian by the definition of D_j, while its projection onto the later summands is 2-Hilbertian by Lemma 3.7 and (4.1). Taking the ℓ_2-sum of these two Hilbert norms shows that the original subspace is D_j-Hilbertian. For every $j \in J$ and every closed subspace F with $E_j \subseteq F \subseteq Z$, we again have $F = E_j \oplus_2 (F \cap W_j)$ isometrically. The same local Hilbert bound therefore applies to $F \cap W_j$. The proof of Theorem 3.9, with these constants, gives

$$\lambda_{\mathrm{DPR}}(Z; E_j) \geq \frac{\Lambda_j^{1/8}}{5} \longrightarrow \infty \qquad (j \in J,\ j \to \infty).$$

Thus every sum of this form fails DPR-lust and has no unconditional basis.

Let $F_j = \ker R_{n_j}$ for $j \geq 1$. Define $Q_j \colon V_j \to V_j$ by $Q_j = R_{n_j}$ for odd j and $Q_j = I_{V_j} - R_{n_j}$ for even j. The required ambient space and projection are

$$X = \left(\bigoplus_{j=1}^{\infty} V_j\right)_2, \qquad Q = \mathrm{diag}(Q_j : j \in \mathbb{N}).$$

Then $\|Q\|, \|I_X - Q\| \leq 1 + \rho/2 < 1 + \rho$. QX is isomorphic to the sum with E_j at the odd indices and F_j at the even indices; for $(I_X - Q)X$ the roles are reversed. Indeed, the operators A_{n_j} on the E_j summands, together with the identities on the F_j summands, define these isomorphisms. Their inverses are given by the restrictions of B_{n_j} and the same identities, so the maps and their inverses have norm at most two. Each of the two sums contains infinitely many of the summands E_j, and hence QX and $(I_X - Q)X$ fail DPR-lust.

The quotient argument of Proposition 4.3 also applies to each of these sums. For each $j \in J$, the sum of the earlier summands has Hilbert distortion at most D_j, and the later summands satisfy the same local estimates. Since J is infinite for each sum, both dual spaces fail DPR-lust.

The space X is a separable uniformly convex Banach lattice with its canonical 1-unconditional basis. Lemma 4.1 and the duality identity used in Corollary 4.4 give, for $R \in \{Q, I_X - Q\}$,

$$\chi_{\mathrm{GL}}(RX) \leq \|R\| < 1 + \rho, \qquad \chi_{\mathrm{GL}}((RX)^*) \leq \|R\| < 1 + \rho.$$

Since all four spaces fail DPR-lust, none is isomorphic to a Banach lattice. This proves (i).

For (ii), observe that all four spaces are superreflexive and are, isometrically, ℓ_2-sums of finite-dimensional spaces. Their block decompositions are therefore contractive and unconditional and yield the metric approximation property, as in Corollary 4.2(iii) and Corollary 4.4(iii). ☐

5. Geometric properties of the examples

We now record some geometric properties of the examples. We show that the spaces and their duals admit conditional asymptotically monotone bases, and conclude with two consequences of their ℓ_2-sum structure.

5.1. **Conditional bases.**

The spaces Z_ρ of Theorem B admit asymptotically monotone bases. In the real construction, the finite sampling projections have Fourier subspaces as both their ranges and their kernels, so we begin with an estimate for the basis constants of such spaces.

Lemma 5.1. *There is an absolute constant $K \geq 1$ such that the following holds. Let $n \geq 1$, $M \geq 3$ and $2 \leq p \leq 4$, and give $G = (\mathbb{Z}/M\mathbb{Z})^n$ its uniform probability measure. Every real Fourier subspace of $L_p(G)$ whose frequency set is invariant under negation has an L_2-orthonormal basis with basis constant at most $K^{2-4/p}$.*

Proof. Order the characters

$$\chi_\nu(r) = \exp\left(\frac{2\pi i}{M} \sum_{s=1}^{n} \nu_s r_s\right), \qquad \nu, r \in \{0, \dots, M-1\}^n,$$

lexicographically by frequency. For $0 \leq m \leq M^n$, let K_m be the set of the first m frequencies and F_m the corresponding Fourier projection. Reversing the coordinate order identifies these with the Vilenkin partial-sum projections on $(\mathbb{Z}/M\mathbb{Z})^{\mathbb{N}}$ restricted to functions of the first n coordinates. Young's uniform L_p estimate [33], recalled in [34, (1.2)], therefore gives $\|F_m\|_{4\to 4} \le C_4$ for an absolute constant $C_4 \ge 1$.

To obtain a real basis, we group frequencies with their negatives, since $\overline{\chi_\nu} = \chi_{-\nu}$. Let $Cf(r) = f(-r)$. This is an isometric involution satisfying $C\chi_\nu = \chi_{-\nu}$, so CF_mC is the Fourier projection onto $-K_m$. Since Fourier projections commute,

$$S_m = F_m + CF_mC - F_mCF_mC$$

is the Fourier projection onto $K_m \cup (-K_m)$: the last term subtracts the projection onto the intersection. This frequency set is invariant under negation, so S_m preserves real-valued functions. Moreover, S_m is orthogonal on L_2 and has L_4 norm at most $2C_4 + C_4^2$.

Order the orbits $\{\nu, -\nu\}$ by their first lexicographic occurrence. For a two-element orbit take $\sqrt{2}\,\mathrm{Re}\,\chi_\nu$ followed by $\sqrt{2}\,\mathrm{Im}\,\chi_\nu$; for a one-element orbit take the real character χ_ν. These vectors form an orthonormal basis of the real space $L_2(G)$. For each two-element orbit, the complex span of these two real vectors is $\mathrm{span}\{\chi_\nu, \chi_{-\nu}\}$. Thus an initial projection ending after a complete orbit is S_m, where m is the position at which that orbit first appears.

It remains to consider an initial projection ending after only the real part of a two-element orbit. If that orbit first appears at position m, this projection is $S_{m-1} + P_u$, where P_u is the orthogonal projection onto $u = \sqrt{2}\,\mathrm{Re}\,\chi_\nu$. Since $P_u f = (\int_G f\overline{u})\, u$ and $|u| \le \sqrt{2}$, Hölder's inequality gives

$$\|P_u\|_{4\to 4} \le \|u\|_4 \|u\|_{4/3} \le 2.$$

Consequently, every initial projection has L_2 norm at most one and L_4 norm at most $K = 2C_4 + C_4^2 + 2$. These estimates hold on the full complex scalar spaces, so interpolation between L_2 and L_4, followed by restriction to real functions, gives the bound $K^{2-4/p}$.

A frequency set invariant under negation selects a subsequence of this real basis. Since the omitted coefficients vanish on the corresponding subspace, its initial projections are restrictions of those of the full basis and satisfy the same bound. $\square$

In Section 3.6, the projection $R_n = A_nB_n$ on U_n has Fourier support $\{-1,1\}^n$, since its matrix kernel, for $r, t \in \{0, \dots, M-1\}^n$, is

$$\left(\frac{2}{M}\right)^n \prod_{s=1}^{n} \cos\left(\frac{2\pi(r_s - t_s)}{M}\right) = \frac{1}{M^n} \sum_{\varepsilon \in \{-1,1\}^n} \exp\left(\frac{2\pi i}{M} \sum_{s=1}^{n} \varepsilon_s (r_s - t_s)\right).$$

Both this set and its complement are invariant under negation. Applying Lemma 5.1 with $M = 1000n^2$ and $p_n = 2 + n^{-3/4}$, we obtain bases of R_nU_n and $\ker R_n$ with constants

$$\beta_n = K^{2-4/p_n} \longrightarrow 1. \tag{5.1}$$

For these choices of M and p_n, bases with constants tending to one can also be obtained from the Lebesgue-constant estimates of Blahota, Persson and Tephnadze [4, Theorem 1], using the same pairing argument and interpolation between L_2 and L_∞.

An analogous ordering is available for the complex sphere spaces of Section 3.1. For $t = (t_s)_{s=1}^n \in \Omega^n$, write $w_{s,k}(t) = w_k(t_s)$ for $1 \le s \le n$ and $k \in \{1,2\}$. Consider the orthonormal monomials

$$h_\iota = \prod_{s=1}^{n} \overline{w_{s,\iota_s}}, \qquad \iota \in \{1,2\}^n.$$

The measure-preserving circle action $w_{s,k} \mapsto e^{-ik3^{s-1}t} w_{s,k}$, for $1 \le s \le n$ and $k \in \{1,2\}$, assigns these monomials the distinct positive frequencies $\sum_{s=1}^{n} \iota_s 3^{s-1} < 3^n$. Ordering them by frequency makes each initial projection a restriction of a spectral interval projection for this action. Such a projection has norm at most one on the full scalar L_2 space and at most $1 + n\log 3$ on L_∞. Indeed, it is given by integration against a Dirichlet kernel, whose normalised L_1 norm for an interval of L frequencies is at most $1 + \log L$. The latter estimate follows by integrating $\min\{L, \pi/|t|\}$ over $[-\pi, \pi]$. Interpolation therefore gives basis constants $(1 + n\log 3)^{1-2/p_n} \to 1$. Under the finite embeddings, these basis constants increase by a factor of at most $c_{p_j}^{n_j}/(1-\delta_j)$ for $j \geq 1$, which tends to one with the choice $(\delta_j)_{j=1}^{\infty} \to 0$ made in Section 3.6.

We now pass from the finite-dimensional bases to asymptotically monotone bases of the examples and their duals.

Corollary 5.2. *The following statements hold for Z_ρ of Theorem B and its dual over either scalar field. Over $\mathbb{R}$, they also hold for QX and $(I_X - Q)X$ of Corollary 4.5 and their duals.*

(i) *Each space admits a conditional basis which is shrinking and boundedly complete.*
(ii) *These bases can be chosen to be asymptotically monotone.*

Proof. For (i), let Z denote Z_ρ, QX or $(I_X - Q)X$, and write $Z = (\bigoplus_{j=1}^{\infty} Z_j)_2$ in its inherited norm. Concatenate the finite bases constructed above. An initial projection acts as the identity on preceding blocks, an initial coefficient projection on the current block, and zero on later blocks. Its norm is therefore bounded by the supremum of the finite basis constants. Reflexivity ensures that these bases are shrinking and boundedly complete [23, Theorem 1.b.5]. The biorthogonal sequences are bases of the duals, which are also reflexive. An unconditional basis of a dual would, by reflexivity, give an unconditional basis of Z, contrary to its construction. All these bases are therefore conditional.

For (ii), an initial projection has norm at most the basis constant of the block in which it ends. Since these constants tend to one and every block is finite-dimensional, the concatenated basis is asymptotically monotone. The same estimate on $(\bigoplus_{j=r}^{\infty} Z_j)_2$ gives basis constants tending to one as $r \to \infty$. The biorthogonal bases have the same initial projection norms, both on the whole space and on the corresponding dual tails, so these conclusions also hold for the duals. □

Given $\varepsilon > 0$, starting the recursive choice sufficiently far out and, over $\mathbb{C}$, taking the embedding errors $(\delta_j)_{j=1}^{\infty}$ sufficiently small makes all these basis constants less than $1 + \varepsilon$, simultaneously for these spaces and their duals. Here the examples are chosen depending on both ρ and ε.

Remark 5.3. The ℓ_2-sum structure of Z_ρ, QX, $(I_X - Q)X$ and their duals ensures that every infinite-dimensional closed subspace contains a copy of ℓ_2 complemented in the whole space; see [25, proof of Proposition 2.1(b) and Proposition 2.4].

The real spaces Z_ρ, QX and $(I_X - Q)X$ also have type 2 and cotype s for every $s > 2$, by the standard L_p estimates and $p_j \to 2$. They fail cotype 2 by Kwapień's theorem [20], since none is isomorphic to a Hilbert space.

AI statement. OpenAI's ChatGPT 5.6 Sol and 6.0 Astra were used during the development of this work.

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